Working with computably Lipschitz reducibility above (uniformly) non-low₂ c.e. degrees

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A Turing degree **d** is non-low₂ if for any total function $f \leq_T \emptyset'$ there is a total function $g \leq_T \mathbf{d}$ which is not dominated by f, i.e., $\exists^{\infty} n[g(n) \geq f(n)].$

A Turing degree **d** is uniformly non-low₂ if there is a computable function *l* such that if $\Phi_e^{\emptyset'}$ is total then $\Phi_{l(e)}^{\mathsf{d}}$ is total and not dominated by it.

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A Turing degree **d** is totally ω -c.e. if every total function $g \leq_T \mathbf{d}$ is ω -c.e..

In the c.e. Turing degrees,

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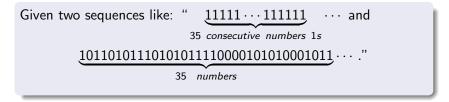
- Let *M* be a Turing machine: $M(\tau) = \sigma = 2^{2^{2^{\tau}}}$. Then τ is an *M*-description of σ .
- For instance, if $\tau = 101$, then

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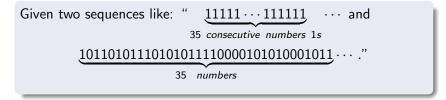
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$$C_{\mathcal{M}}(\sigma) = \min\{|\tau|, \infty : \mathcal{M}(\tau) = \sigma\},\$$

- For a universal machine U, $C(\sigma) = C_U(\sigma) \le C_M(\sigma) + O(1)$.
- In randomness and incomputability we have two fundamental measures: the plain complexity C and the prefix-free complexity K.
- Given M and U are prefix-free, K_M(σ) and K(σ) = K_U(σ) are well-defined.

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A real α is 1-random if $\forall n[K(\alpha \upharpoonright n) > n - c]$.

Definition (Martin-Löf, P., 1966)

A real α is Martin-Löf random if for all computable collections of c.e. open sets $\{U_n : n \in \omega\}$, with $\mu(U_n) \leq 2^{-n}, \alpha \notin \bigcap_n U_n$.

Theorem (Schnorr, 1973)

- The following are equivalent for a real α .
 - α is 1-random;
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- Real α is Δ⁰₂ (left-c.e.) if it is the limit of a computable (increasing) sequence of rational numbers.
- For a universal prefix-free machine U, $\Omega_U = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}$ is a left-c e random real
- $\alpha \leq_{\kappa} \beta$ if $K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + O(1)$.
- $\alpha \leq_C \beta$ if $C(\alpha \upharpoonright n) \leq C(\beta \upharpoonright n) + O(1)$.
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- (Soare,2013) The identity bound Turing reducibility (ibT).
- If $\alpha \leq_{cl} \beta$, then for all n, $K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + O(1)$.
- The cl-degree only contains either only random reals or non-random reals.
- (Downey, Hirschfeldt, Lafort 2001) The cl-degrees of left-c.e. reals is neither a lower semi-lattice, nor an upper semi-lattice.

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- (Yu and Ding,2004) There are two c.e.reals α and β which have no common upper bound under cl-reducibility in left-c.e. reals.
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For a c.e. degree **d**, the following are equivalent:

(1) **d** is array non-computable.

(2) There is a cl-maximal pair of left-c.e. reals (α, β) in **d**.

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- (α, β) is a cl-maximal pair of left-c.e. reals if no left-c.e. real can cl-compute both of them.
- (*A*, *B*) is a cl-maximal pair of c.e. sets if no c.e. set can cl-compute both of them.
- (Barmpalias,2005; Fan and Lu,2005) There exists a cl-maximal pair of c.e. sets.

Theorem (Ambos-spies, Ding, Fan and Wolfgang, 2013)

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- (Kjos-Hanssen, Wolfgang, Stephen, 2006) A set A is complex if there is an order (nondecreasing, unbounded, computable) function h such that $K(A \upharpoonright x) > h(x)$ for all x.
- (Downey, Hirschfeldt, 2004) There is a real (not c.e.) which is not cl-reducible to any random real (indeed to any complex real).
- A degree **d** is called generalised low₂ if $\mathbf{d}'' \leq (\mathbf{d} \vee \mathbf{0})'$.
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(1) **d** is not totally ω -c.e..

(2) There is a left-c.e. real β in **d** which is not cl-reducible to any complex left-c.e. real.

Uniformly non-low₂-ness and Computably Lipschitz reducibility

Theorem (Fan, 2017)

If a c.e. degree \boldsymbol{d} is uniformly non-low_2, then

for any non-computable Δ_2^0 real α , there is a left-c.e. real β in **d** such that both of them have no common upper bound of c.e. reals under cl-reducibility.

Theorem (Fan, unpublished)

If a c.e. degree **d** is uniformly non-low₂, then

for any non-computable Δ_2^0 real α , if $\alpha \leq_T \gamma$ for the left-c.e. real γ , then there is a left-c.e. real $\beta \leq_T d$ such that $\beta \leq_{cl} \gamma$.

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- (Kjos-Hanssen, Wolfgang, Stephen, 2006) A set A is auto-complex if there is a nondecreasing, unbounded,total function h ≤_T A such that K(A ↾ x) > h(x) for all x.
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Suppose that \boldsymbol{d} is a non-low_2 c.e. degree, then

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There is no cl-maximal c.e set.

Theorem (Lewis,Barmpalias,2007)

- There exists a quasi-maximal cl-degree, i.e. there exits a real α , such that, for all reals β , if $\alpha \leq_{cl} \beta$, then $\beta \leq_{T} \alpha$. In fact, every random real satisfies the quasi-maximality property.
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Thank you!



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