On one-variable modal μ-calculus Part 2

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Introduction

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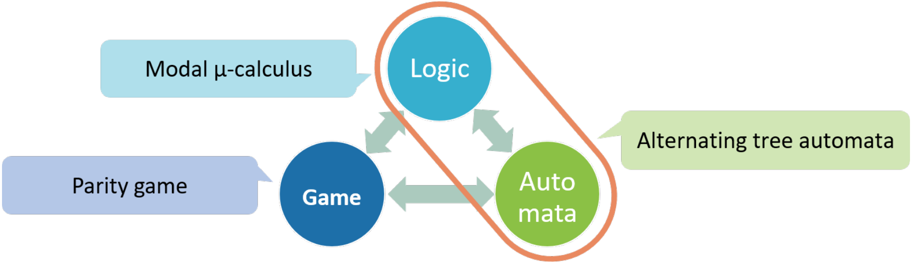
1. On the correspondence between one-variable Modal μ-calculus and weak alternating tree automata,
2. Prove the strictness of alternation hierarchy of one-variable modal μ-calculus.
3. Introduce a transfinite extension of weak parity games.



Alternating tree automata

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Definition (Alternating tree automata)

An *Alternating tree automaton (ATA)* is a tuple $\mathcal{A} = (Q, q_I, \delta, \Omega)$ where

- Q is a finite set of states,
- $q_I \in Q$ is a state called an initial state,
- $\delta : Q \to TC^Q$ is called a transition function,
- $\Omega : Q \to \omega$ is called a priority function.

TC^Q, transition condition over Q is defined by:

- $\bot, \top \in TC^Q$,
- $p, \neg p \in TC^Q$, for every $p \in P$,
- $q, \Box q, \Diamond q \in TC^Q$, for every $q \in Q$,
- $q \vee q', q \wedge q' \in TC^Q$, for $q, q' \in Q$.

Alternating tree automata

Definition

The computational behavior of alternating tree automata is explained using the notion of a run.

Definition (run of alternating tree automata)

A *run* of $\mathcal{A}$ on $(\mathbb{S}, s)$ is a $(S \times Q)$-vertex labeled tree $R = (V^R, E^R, \lambda^R)$ such that for every vertex v with label (s, q) the following conditions are satisfied:

- $\delta(q) \neq \bot$,
- $\delta(q) = p \Rightarrow s \in \lambda(p)$, and $\delta(q) = \neg q \Rightarrow s \notin \lambda(p)$ for $p \in P$,
- $\delta(q) = q' \Rightarrow \exists v' \in S(v)$ s.t. $\lambda(v') = (s, q')$ for $q' \in Q$,
- $\delta(q) = q' \vee q'' \Rightarrow \exists v' \in S(v)$ s.t. $\lambda(v') = (s, q')$ or (s, q''),
- The same applies below.

A run is *accepting* if the state labeling of every infinite branch through R satisfies the parity acceptance condition determined by Ω.

Alternating tree automata

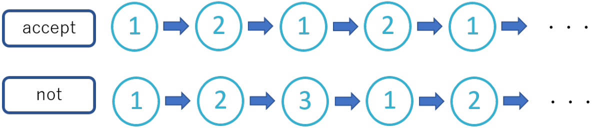
Definition

We define parity condition of run of the infinite path.

Definition (Parity condition)

A path π satisfies the *parity condition* if the following holds:

$$max\{\Omega(q) \mid q \in Inf(\pi)\} \text{is even.}$$

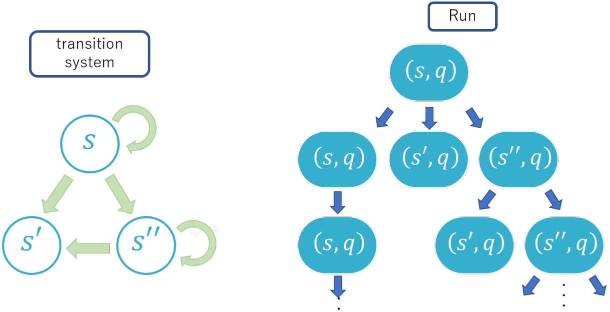


For the finite path, the path is accepted if p is true in state q, otherwise not accepted (in definition, there is no run containing such a path).

Example

Consider $\mathcal{A} = (\{q\}, q, \delta(q) = \Box q, \Omega(q) = 1)$.

$$\mathcal{A} \text{ accepts } (\mathcal{S}, s) \iff \text{All paths starting from } s \text{ are finite.}$$



The pointed transition system accepted by this $\mathcal{A}$ will be all paths starting from s which are finite only.

Correspondence between modal μ-calculus and alternating tree automata

Theorem

There is an effective translation procedure between an L_μ-formula φ and an alternating tree automaton $\mathcal{A}$ so that for all finitely branching transition system $(\mathcal{S}, s)$,

$$(\mathcal{S}, s) \models \varphi \iff (\mathcal{S}, s) \in L(\mathcal{A}).$$

Example

For example, $\varphi = \mu p.\Box p$ and $\mathcal{A} = (\{q\}, q, \delta(q) = \Box q, \Omega(q) = 1)$.
These two are equivalent.

One-variable L_μ-formula and Weak ATA

Definitions

Consider restricted to one variable L_μ-formulas, automata and parity games with a condition added to the priority function.

Definition (n-variable L_μ-formula)

For any n, we denote by $L_\mu[n]$ the set of L_μ formulas that have at most n distinct variables bounded by μ and ν.
A formula of $L_\mu[n]$ is called an n-variable L_μ-formula.

Definition (Weak alternating tree automata)

An ATA $\mathcal{A} = (Q, q_I, \delta, \Omega)$ with a priority function $\Omega : Q \to \{0, \cdots, n\}$ is said to be *weak* if δ has the following additional property :

$$\text{for all } q \in Q, \text{ if } q' \text{ occurs in } \delta(q), \text{ then } \Omega(q') \leq \Omega(q).$$

One-variable L_μ-formula and Weak ATA

Then, we have the following theorem:

Main theorem

There is an effective translation procedure between a one-variable L_μ-formula φ and a weak alternating tree automaton $\mathcal{A}$ so that for all finitely branching transition system $(\mathcal{S}, s)$,

$$(\mathcal{S}, s) \models \varphi \iff (\mathcal{S}, s) \in L(\mathcal{A}).$$

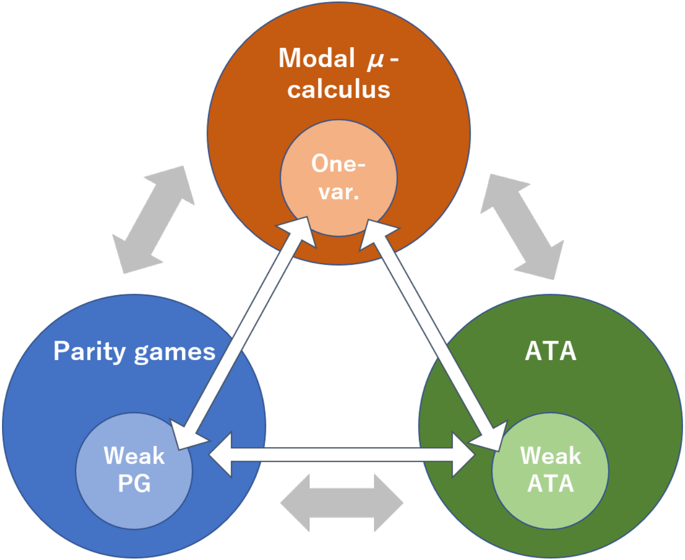
One-variable L_μ-formula and Weak ATA

Furthermore,

Corollary

The alternation depth of a one-variable L_μ-formula corresponds to the number of the priorities of the associated automaton.

Correspondence of three models



The strictness of alternation hierarchy of one-variable modal μ-calculus

Next, we consider the relations of the hierarchies of one-variable L_μ-formulas, weak alternating tree automata and weak parity game.

Parity games

A *parity game* $\mathcal{G} = (V_0, V_1, E, \Omega)$ is played on a colored directed graph, where each node is colored by the priority function Ω. Two players, P0 and P1, move a token along the edges of the graph which results in a path, called the play. For any position $v \in V_0 \cup V_1$, if $v \in V_0$ (V_1), P0 (P1) chooses a successor v' such that $(v, v') \in E$.

Definition (Weak parity games)

A parity game $\mathcal{P} = (V_0, V_1, E, \Omega)$ is said to be *weak* if Ω has the following additional property :

$$\text{for all } v, v' \in V_0 \cup V_1, \text{ if } (v, v') \in E, \text{ then } \Omega(v) \leq \Omega(v').$$

Strictness of alternating hierarchy of one-variable modal μ-calculus

The next theorem is first proved by Mostwski. Here we introduce an easier formula to show the strictness of hierarchy for one-variable L_μ-formulas.

theorem

Alternation hierarchy for one-variable L_μ-formulas is strictness.

Strictness of alternating hierarchy of one-variable modal μ-calculus

theorem

Alternation hierarchy for one-variable L_μ-formulas is strictness.

Sketch of roof) Consider the recursively defined φ_n^w as follows:

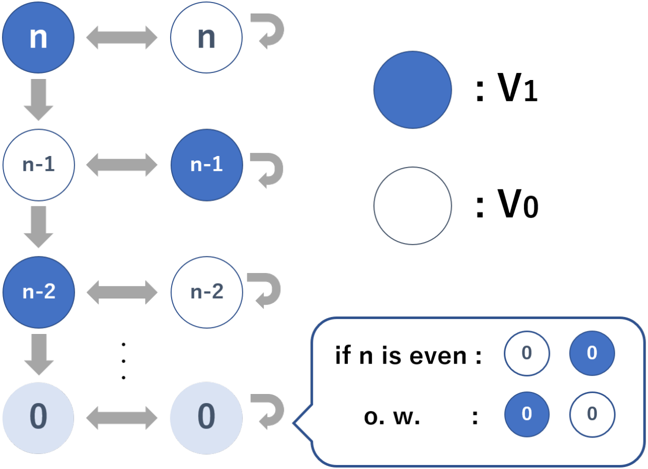
- $\varphi_0^w = \nu x\{(p \wedge p_0' \wedge \Diamond x) \vee (\neg p \wedge p_0' \wedge \Box x)\}$,
- $\varphi_n^w = \eta x\{(p \wedge p_n' \wedge \Diamond x) \vee (\neg p \wedge p_n' \wedge \Box x) \vee \varphi_{n-1}^w\}$ $(n > 0)$,

where, η is ν if n is even, otherwise μ.

This formula means the existence of a winning strategy for P0 at weak parity games, also witness the strictness the in the alternation hierarchy for one-variable modal μ-calculus.

Transfinite extension of weak parity games

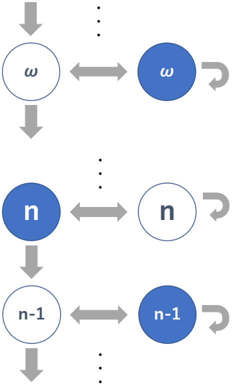
First, we consider the following weak parity game $\mathcal{P}_n$:



This game needs n priorities. (can not write with $n-1$ or less priorities.)

Transfinite extension of weak parity games

Now, we consider the following weak parity game $\mathcal{P}_\omega$:



This game needs ω priorities. (can not write with infinite priorities.)

Transfinite extension of weak parity games

In general, the transfinite extension of weak parity games, that is weak parity games with ordinal priorities.
For an ordinal ω, we define $\mathcal{P}_\omega = (V_0, V_1, E, \Omega)$ as follows:

- $V_0 = \{v_i, v'_{i+1} \mid \text{for all even number } i \in \omega\}$,
- $V_1 = \{v'_i, v_{i+1} \mid \text{for all even number } i \in \omega\}$,
- $E = \{(v_i, v_{i-1}) \mid 1 \leq i < \omega\} \cup \{(v_i, v'_i), (v'_i, v'_i) \mid 0 \leq i < \omega\} \cup \{(v_\omega, v_i) \mid i < \omega\}$,
- $\Omega(v_i) = \Omega(v'_i) = i \ (0 \leq i < \omega), \Omega(v_\omega) = \omega$.

Question: In this game, infinite number of priorities are required for non-weak parity games ?

Transfinite extension of weak parity games

This game can be expressed with a finite number of priority, in particular two-priority games, by using a non-weak parity game.
In general, the following holds:

theorem

Any transfinite weak parity game can be expressed with two-priority parity game. In other words, For any transfinite weak parity game $\mathcal{P}$, there is a two-priority parity game $\mathcal{P}'$ such that $Win_{P0}(\mathcal{P}) = Win_{P0}(\mathcal{P}')$.

Proof.

Consider a rewritten weak parity game of ω-priority as follows: change all even priority to 0, and all odd priority to 1. Then, the game becomes a two-priority parity game equal to the original game. In the ω-priority parity game, since it is the weak, only one priority of infinitely occurs. Therefore, the priority is used only by discriminating between even and odd, so we can write the game equivalent to original with priority 0 and 1 only. □

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Thank you so much for your kind attention.