Ramsey Typed Theorems and Reverse Mathematics

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Ramsey's Theorem

For $A \subseteq \mathbb{N}$, let $[A]^n$ denote the set of all n-element subsets of A.

Theorem (Ramsey 1930)

Any $f : [\mathbb{N}]^n \to \{0, 1, \ldots, k-1\}$ *has an infinite* homogeneous set $H \subseteq \mathbb{N}$, *namely,* f *is constant on* $[H]^n$.

Loosely speaking: Every coloring problem has a homogeneous solution.

Notation: The version above is denoted by RT^n_k.

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One Proof of RT^2_2

Let f be a coloring of pairs, say by red and blue.

- First step: Find an infinite subset $C \subseteq \mathbb{N}$ on which f is "stable", i.e., for all x, $\lim_{y \in C, y \to \infty} f(x, y)$ exists.
- We call such a set C *cohesive* for f.
- Second step: One of
  $D^R = \{x \in C : x \text{ is "eventually red"}\}$ and
  $D^B = \{x \in C : x \text{ is "eventually blue"}\}$
  must be infinite, say D^R.
- Obtain a solution from D^R.

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COH and SRT^2_2

We extract two combinatorial principles out of the proof:

- Let R be an infinite set and $R^s = \{t | (s,t) \in R\}$. A set G is said to be *R-cohesive* if for all s, either $G \cap R^s$ is finite or $G \cap \overline{R^s}$ is finite.
- The cohesive principle COH states that for every R, there is an infinite G that is R-cohesive.
- SRT^2_2 states that every *stable* coloring of pairs has a solution.
- (Cholak, Jockusch and Slaman, 2001)

$$\mathrm{RT}^2_2 = \mathrm{COH} + \mathrm{SRT}^2_2.$$

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Motivating Questions

- How complicated is the homogeneous set H?
- Is COH or SRT^2_2 as strong as RT^2_2?
- What are the logical consequences/strength of Ramsey's Theorem?
- We need to introduce "measures" of the strengths.

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Measuring Strength

- The way to show that $P \not\Rightarrow Q$ is to “make” P true and Q false. But these combinatorial principles are all true.
- Thus we have to work in some weaker systems Γ, and demonstrate that “Γ proves P but not Q”.
- Often we will have a hierarchy of systems $\Gamma_0 < \Gamma_1 < \ldots$, and Γ_i proves P but not Q (or better, Q proves Γ_j for some $j > i$).
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Fragments of First Order Peano Arithmetic

- Let $I\Sigma_n$ denote the induction schema for Σ^0_n-formulas; and $B\Sigma_n$ denote the Bounding Principle for Σ^0_n formulas.
- (Kirby and Paris, 1977) $\cdots \Rightarrow I\Sigma_{n+1} \Rightarrow B\Sigma_{n+1} \Rightarrow I\Sigma_n \Rightarrow \ldots$
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Big Five in Reverse Mathematics

- Reverse mathematics uses fragments of Second Order Arithmetic.
- RCA_0: Σ^0_1-induction and Δ^0_1-comprehension:
  For $\varphi \in \Delta^0_1$, $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$.
- WKL_0: RCA_0 and every infinite binary tree has an infinite path.
- ACA_0: RCA_0 and for φ arithmetical, $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$.
- (ATR_0 and Π^1_1-CA_0.)

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Remarks on Goals of Reversion

- Goal of Reverse Mathematics: What set existence axioms are needed to prove the theorems of ordinary, classical (countable) mathematics?
- Goal of Reverse Recursion Theory: What amount of induction are needed to prove the theorems of Recursion Theory, in particular, theorems about r.e. degrees.
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Rephrasing the Motivating Questions

- Question: Suppose *f* is recursive. What is the minimal syntactical complexity of a solution?
- Question: Which system in Reverse Mathematics does Ramsey's Theorem correspond? E.g., does RT^2_2 imply ACA_0?
- What are the first-order consequences of Ramsey's Theorem? E.g., does RT^2_2 imply $I\Sigma_2$?
- Does SRT^2_2 imply RT^2_2? In other words, if $\mathcal{X}$ contains solutions for all stable colorings, how about for general colorings? (Here $\mathcal{X}$ is the second order part of the model.)

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Earlier Results: (I)

Theorem (Jockusch 1972)

Over RCA_0,

$$\mathsf{ACA}_0 \Leftrightarrow \mathsf{RT}^3_2 \Leftrightarrow \mathsf{RT}^n_k.$$

$$\mathsf{ACA}_0 \Rightarrow \mathsf{RT}^2_2 \quad \textit{and} \quad \mathsf{WKL}_0 \not\Rightarrow \mathsf{RT}^2_2.$$

Theorem (Hirst 1987)

Over RCA_0,

$$(\mathsf{S})\mathsf{RT}^2_2 \Rightarrow B\Sigma_2.$$

Theorem (Seetapun and Slaman 1995)

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Conservation Results

- Harrington observed that WKL_0 is Π^1_1-conservative over RCA_0. i.e., any Π^1_1-statement that is provable in WKL_0 is already provable in RCA_0.
- Conservation results are used to measure the weakness of the strength of a theorem.

Theorem (Cholak, Jochusch and Slaman 2001)
RT^2_2 *is* Π^1_1*-conservative over* $\mathsf{RCA}_0 + I\Sigma^0_2$*, Hence,*

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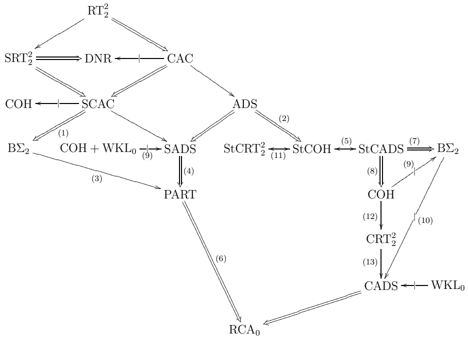
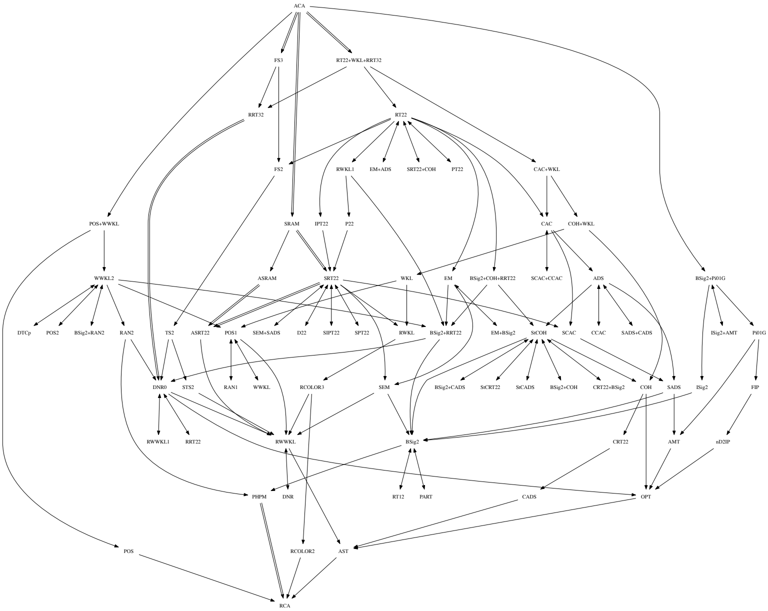
Combinatorics below RT^2_2

Hirschfeldt and Shore [2007], *Combinatorial principles weaker than Ramsey's theorem for pairs*.



In particular, COH does not imply RT^2_2.

From Big Five to The Zoo



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Remarks

- One approach to show $\mathsf{SRT}^2_2 \not\Rightarrow \mathsf{RT}^2_2$: Show that stable colorings always have a low solution. Or equivalently, every Δ^0_2-set contains or is disjoint from an infinite low set.

- Downey, Hirschfeldt, Lempp and Solomon (2001): There is a Δ^0_2 set D such that neither D nor $\overline{D}$ contains infinite low subset.

- Chong (2005): We should look at nonstandard models of fragments of arithmetic, because:
  - DHLS theorem is done on ω, whose proof involves infinite injury method thus requires $I\Sigma^0_2$.
  - There is a model of $B\Sigma^0_2$ but not $I\Sigma^0_2$ in which every incomplete Δ^0_2 set is low.

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- What happens in ω-model? (Kind of "provability vs. truth" question.)
- How about conservation results? E.g., Is RT^2_2 or SRT^2_2 Π^1_1-conservative over RCA_0?
- (Downey and Ng) Can we improve DHLS Theorem to low_2 and Δ^0_2 sets?
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- Let $T = 2^{<\omega}$ be the full binary tree.
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(with Chitat Chong, Wei Li and Jiayi Liu)

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