Π^1_n-indescribabilities in proof theory

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In this talk let us report a recent proof-theoretic reduction on indescribable cardinals.

It is shown that over $\mathsf{ZF}+(V=L)$, the existence of a Π^1_1-indescribable cardinal is proof-theoretically reducible to iterations of Mostowski collapsings and Mahlo operations. The same holds for Π^1_{n+1}-indescribable cardinals and Π^1_n-indescribabilities.

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1 Indescribable cardinals

Consider the language $\{\in, R\}$ with a unary predicate symbol R. Π^1_0 denotes the set of first-order formulas in the language $\{\in, R\}$, and Π^1_n the set of second-order formulas $\forall X_1 \exists X_2 \cdots Q X_n \, \varphi$.

Definition 1.1 [Hanf-Scott61]
For $n \geq 0$, a cardinal κ is said to be Π^1_n-*indescribable* iff for any $A \subset V_\kappa$ and any Π^1_n-sentence $\varphi(R)$, if $V_\kappa \models \varphi[A]$, then $V_\alpha \models \varphi[A \cap V_\alpha]$ for some $\alpha < \kappa$.

Definition 1.2 $S \subset Ord$ is said to be Π^1_n-*indescribable in* κ iff for any $A \subset V_\kappa$ and any Π^1_n-sentence $\varphi(R)$, if $V_\kappa \models \varphi[A]$, then $V_\alpha \models \varphi[A \cap V_\alpha]$ for some $\alpha \in S \cap \kappa$.

Facts.

1. A cardinal is inaccessible(, i.e., regular and strong limit) iff it is Π^1_0-indescribable.
2. For regular uncountable κ, S is Π^1_0-indescribable in κ iff S is stationary in κ, i.e., S meets every club (closed and unbounded) subset of κ.
3. [Hanf-Scott61] A cardinal is Π^1_1-indescribable iff it is weakly compact, i.e., inaccessible and has the tree property.

   By definition, κ has the tree property if every tree of height κ whose levels have size less than κ has a branch of length κ.

Let Rg denote the class of regular uncountable cardinals, and $S \subset Ord$.

Definition 1.3 (Mahlo operation)

$$\begin{aligned} M_0(S) &:= \{\sigma \in Rg : S \text{ is stationary in } \sigma\} \\ &= \{\sigma \in Rg : S \text{ is } \Pi^1_0\text{-indescribable in } \sigma\} \end{aligned}$$

Definition 1.4 For $n \geq 0$,

$$M_n(S) := \{\sigma \in Rg : S \text{ is } \Pi^1_n\text{-indescribable in } \sigma\}.$$

Lemma 1.5

$$M_{n+1}(Ord) \cap M_n(S) \subset M_n(M_n(S)).$$

Namely if κ is a Π^1_{n+1}-indescribable cardinal and $S \subset Ord$ is Π^1_n-indescribable in κ, then $M_n(S)$ is Π^1_n-indescribable in κ.

Proof. This follows from the fact that there exists a Π^1_{n+1}-sentence $m_n(S)$ such that $\kappa \in M_n(S)$ iff $V_\kappa \models m_n(S)$, which in turn follows from the existence of a universal Π^1_n-formula. $\square$

Hence if $\kappa \in M_{n+1}(Ord) = M^1_{n+1}$, then

$$\kappa \in M_n(M_n(Ord)) = M^2_n, M^3_n, \ldots, M^\alpha_n \, (\alpha < \kappa), M^\triangle_n, \ldots$$

where $\kappa \in M^\triangle_n :\Leftrightarrow \kappa \in \bigcap_{\alpha<\kappa} M^\alpha_n$.

Actually Lemma 1.5 characterizes, over $V = L$, the weak compactness of regular uncountable cardinals κ.

Theorem 1.6 [Jensen72] Assume $V = L$. For regular uncountable cardinals κ,

$$\begin{aligned}\kappa \in M_1^1 &\Leftrightarrow \forall S \subset \kappa[\kappa \in M_0(S) \to Rg \cap M_0(S) \cap \kappa \neq \emptyset] \\ &\Leftrightarrow \forall S \subset \kappa[\kappa \in M_0(S) \to \kappa \in M_0(M_0(S))]\end{aligned}$$

Theorem 1.7 [Bagaria-Magidor-Sakai∞] Assume $V = L$. For Π_n^1-indescribable cardinals $\kappa \in M_n^1$,

$$\begin{aligned}\kappa \in M_{n+1}^1 &\Leftrightarrow \forall S \subset \kappa[\kappa \in M_n(S) \to M_n^1 \cap M_n(S) \cap \kappa \neq \emptyset] \\ &\Leftrightarrow \forall S \subset \kappa[\kappa \in M_n(S) \to \kappa \in M_n(M_n(S))]\end{aligned}$$

Definition 1.8 Let κ be a regular uncountable cardinal.

1. S is (-1)-*stationary* in κ iff $S \cap \kappa$ is unbounded in κ.
2. λ is Π^1_{-1}-*indescribable* iff λ is a limit ordinal.
3. For $n \geq 0$, S is n-*stationary* in κ iff S meets every n-club subset of κ.
4. C is $(n+1)$-*club* in κ iff
   (a) C is n-stationary in κ, and
   (b) if C is n-stationary in Π^1_n-indescribable $\lambda < \kappa$, then $\lambda \in C$.

Let M^1_0 denotes the class of inaccessible cardinals.

Proposition 1.9 [Bagaria-Magidor-Sakai∞]
For $n \geq 0$ and $\kappa \in M^1_n$,

$$\kappa \in M_n(S) \text{ iff } S \text{ is } n\text{-stationary in } \kappa.$$

Corollary 1.10 [Bagaria-Magidor-Sakai∞] Assume $V = L$. For $n \geq 0$ and $\kappa \in M^1_n$, $\kappa \in M^1_{n+1}$ iff

$$\forall S \subset \kappa[S \text{ is } n\text{-stationary in } \kappa \Rightarrow$$
$$\exists \lambda \in M^1_n \cap \kappa(S \text{ is } n\text{-stationary in } \lambda)]$$

2 Reduction of Π^1_{N+1}-indescribability

We now ask:

How far can we iterate the operation M_n of Π^1_n-indescribability in Π^1_{n+1}-indescribable cardinals?

Or proof-theoretically:

Over ZF(ZF+(V=L)), the existence of a Π^1_{n+1}-indescribable cardinal is reducible to iterations of M_n?

Let $<^\varepsilon$ be a Δ-predicate such that for any transitive and well-founded model V of $\mathsf{KP}\omega$, $<^\varepsilon$ is a canonical well ordering of type ε_{I+1} for the order type I of the class Ord of ordinals in V.

I will show that the assumption of the Π^1_{N+1}-indescribability is proof-theoretically reducible to iterations of an operation along initial segments of $<^\varepsilon$ over ZF+(V=L). The operation is a mixture $Mh^\alpha_{N,n}[\Theta]$ of the operation M_N of Π^1_N-indescribability and Mostowski collapsings.

To define the class $Mh^\alpha_{N,n}[\Theta]$, we need first to introduce ordinals for analyzing ZF+(V=L) proof-theoretically in [A∞1].

Let I be a weakly inaccessible cardinal, and L_I the set of constructible sets of L-rank$< I$.

2.1 Skolem hulls and ZF+(V=L)-provable countable ordinals

Definition 2.1 For $X \subset L_I$, $\mathrm{Hull}_{\Sigma_n}(X)$ denotes the Σ_n-*Skolem hull* of X in L_I. $a \in \mathrm{Hull}_{\Sigma_n}(X) \Leftrightarrow \{a\} \in \Sigma_n^{L_I}(X)\,(a \in L_I)$.

Definition 2.2 (Mostowski collapsing function F)

By the Condensation Lemma we have an isomorphism (Mostowski collapsing function)

$$F : \mathrm{Hull}_{\Sigma_n}(X) \leftrightarrow L_\gamma$$

for an ordinal $\gamma \leq I$ such that $F \restriction Y = id \restriction Y$ for any transitive $Y \subset \mathrm{Hull}_{\Sigma_n}(X)$.

Though $I \notin dom(F) = \mathrm{Hull}_{\Sigma_n}(X)$ write

$$F(I) := \gamma.$$

Let us denote the isomorphism F on $\mathrm{Hull}_{\Sigma_n}(X) \leftrightarrow L_\gamma$ by $F^{\Sigma_n}_X$.

Given an integer n, let us define a Skolem hull $\mathcal{H}_{\alpha,n}(X)$ and ordinals $\Psi_{\kappa,n}\alpha$ (regular $\kappa \leq I$) simultaneously by recursion on $\alpha < \varepsilon_{I+1}$, the next ε-number above I.

Definition 2.3 $\mathcal{H}_{\alpha,n}(X)$ is a Skolem hull of $\{0, I\} \cup X$ under the functions $+, \alpha \mapsto \omega^\alpha$, $\Psi_{\kappa,n} \restriction \alpha$ (regular $\kappa \leq I$),
the Σ_n-definability:

$$Y \mapsto \mathrm{Hull}_{\Sigma_n}(Y \cap I)$$

and the Mostowski collapsing functions

$$(x = \Psi_{\kappa,n}\gamma, \delta) \mapsto F^{\Sigma_1}_{x\cup\{\kappa\}}(\delta)\, (\kappa \in Rg\cap I), (x = \Psi_{I,n}\gamma, \delta) \mapsto F^{\Sigma_n}_{x}(\delta).$$

For $\kappa \leq I$

$$\Psi_{\kappa,n}\alpha := \min\{\beta \leq \kappa : \kappa \in \mathcal{H}_{\alpha,n}(\beta)\, \&\, \mathcal{H}_{\alpha,n}(\beta) \cap \kappa \subset \beta\}.$$

For each $\alpha < \varepsilon_{I+1}$, $\mathsf{ZF} + (V = L) \vdash \Psi_{\kappa,n}\alpha < \kappa$.

Theorem 2.4 ([A∞1])

For a sentence $\exists x < \omega_1 \, \varphi(x)$ with a first-order formula $\varphi(x)$, if

$$\mathsf{ZF} + (V = L) \vdash \exists x < \omega_1 \, \varphi(x)$$

then

$$\exists n < \omega [\mathsf{ZF} + (V = L) \vdash \exists x < \Psi_{\omega_1, n} \omega_n (I + 1) \varphi(x)].$$

Thus the countable ordinal

$$\Psi_{\omega_1} \varepsilon_{I+1} := \sup\{\Psi_{\omega_1, n} \omega_n (I + 1) : n < \omega\}$$

is the limit of $\mathsf{ZF} + (V = L)$-provably countable ordinals.

Our proof of Theorem 2.4 is based on ordinal analysis(cut-elimination in terms of operator controlled derivations in [Buchholz92]) and the following observation.

Proposition 2.5 Let $\omega \leq \alpha < \kappa < I$ with α a multiplicative principal number. Then $L_I \models \alpha < cf(\kappa)$ iff there exists an ordinal β between α and κ such that
$\mathrm{Hull}_{\Sigma_1}(\beta \cup \{\kappa\}) \cap \kappa \subset \beta (\Leftrightarrow \beta = F^{\Sigma_1}_{\beta\cup\{\kappa\}}(\kappa))$ and $F^{\Sigma_1}_{\beta\cup\{\kappa\}}(I) < \kappa$.

2.2 The class $Mh^{\alpha}_{N,n}[\Theta]$

In what follows $\mathcal{K}$ denotes a Π^1_{N+1}-indescribable cardinal, and I the least weakly inaccessible cardinal above $\mathcal{K}$. The operator $\mathcal{H}_{\alpha,n}(X)$ is defined as above augmented with $\mathcal{K} \in \mathcal{H}_{\alpha,n}(X)$.

In the following definition, α can be much larger than π.

Definition 2.6 Let $\alpha < \varepsilon_{I+1}$, $\Theta \subset_{fin} (\mathcal{K}+1)$ and $\mathcal{K} \geq \pi$ be regular uncountable. Then $\pi \in Mh^{\alpha}_{N,n}[\Theta]$ iff

$$\begin{aligned} & \mathcal{H}_{\alpha,n}(\pi) \cap \mathcal{K} \subset \pi \,\&\, \alpha \in \mathcal{H}_{\alpha,n}[\Theta](\pi) \\ \& \quad & \forall \xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\pi\}](\pi) \cap \alpha [\pi \in M_N(Mh^{\xi}_{N,n}[\Theta \cup \{\pi\}])] \end{aligned}$$

Roughly $\{\pi\}$ in $\xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\pi\}](\pi)$ allows to define ξ from the point π.

For the case $N = 1$, i.e., Π^1_1-indescribable cardinal $\mathcal{K}$, let us examine the strength of the assumptions $\mathcal{K} \in Mh^{\mathcal{K}+1}_{0,n}[\emptyset]$.

$M^\alpha\,(\alpha < \mathcal{K}^+)$ denotes the set of α-weakly Mahlo cardinals defined as follows. $M^0 := Rg \cap \mathcal{K}$, $M^{\alpha+1} = M_0(M^\alpha)$, $M^\lambda = \bigcap\{M_0(M^\alpha) : \alpha < \lambda\}$ for limit ordinals λ with $cf(\lambda) < \mathcal{K}$, and $M^\lambda := \triangle\{M_0(M^{\lambda_i}) : i < \mathcal{K}\}$ for limit ordinals λ with $cf(\lambda) = \mathcal{K}$, where $\sup_{i<\mathcal{K}} \lambda_i = \lambda$ and the sequence $\{\lambda_i\}_{i<\mathcal{K}}$ is chosen so that it is the $<_L$-minimal such sequence.

In the last case for $\pi < \mathcal{K}$, $\pi \in M^\lambda \Leftrightarrow \forall i < \pi(\pi \in M_0(M^{\lambda_i}))$.

Proposition 2.7 For $n \geq 1$ and $\sigma \leq \mathcal{K}$, the followings are provable in $\mathsf{ZF} + (V = L)$.

1. If $\sigma \in \Theta$, $\pi \in Mh^{\alpha}_{0,n}[\Theta] \cap \sigma$, and $\alpha \in \mathrm{Hull}_{\Sigma_1}(\{\sigma, \sigma^+\} \cup \pi) \cap \sigma^+$, then $\pi \in M^{\alpha}$.

2. If $\sigma \in Mh^{\sigma^+}_{0,n}[\Theta]$, then $\forall \alpha < \sigma^+(\sigma \in M_0(M^{\alpha}))$, i.e., σ is a greatly Mahlo cardinal in the sense of [Baumgartner-Taylor-Wagon77].

3. The class of the greatly Mahlo cardinals below $\mathcal{K}$ is stationary in $\mathcal{K}$ if $\mathcal{K} \in Mh^{\mathcal{K}+1}_{0,n}[\emptyset]$.

Proof. 2.7.3 follows from 2.7.2.

Proof.

2.7.1 by induction on $\alpha < \sigma^+$, show

> If $\sigma \in \Theta$, $\pi \in Mh^{\alpha}_{0,n}[\Theta] \cap \sigma$, and $\alpha \in \mathrm{Hull}_{\Sigma_1}(\{\sigma, \sigma^+\} \cup \pi) \cap \sigma^+$, then $\pi \in M^{\alpha}$.

2.7.2. If $\sigma \in Mh^{\sigma^+}_{0,n}[\Theta]$, then $\forall \alpha < \sigma^+ (\sigma \in M_0(M^{\alpha}))$.

Suppose $\exists \alpha < \sigma^+ (\sigma \not\in M_0(M^{\alpha}))$. Let $\alpha < \sigma^+$ be the minimal such ordinal, and C be a club subset of σ such that $C \cap M^{\alpha} = \emptyset$. $\alpha \in \mathrm{Hull}_{\Sigma_1}(\{\sigma, \sigma^+\}) \cap \sigma^+ \subset \mathcal{H}_{\alpha,n}[\Theta \cup \{\sigma\}](\sigma) \cap \sigma^+$. By $\sigma \in Mh^{\sigma^+}_{0,n}[\Theta]$ we have $\sigma \in M_0(Mh^{\alpha}_{0,n}[\Theta \cup \{\sigma\}])$. Pick a $\pi \in C \cap Mh^{\alpha}_{0,n}[\Theta \cup \{\sigma\}] \cap \sigma$. Proposition 2.7.1 yields $\pi \in M^{\alpha}$. A contradiction. $\square$

Theorem 2.8 ([A∞3], [A∞4])

1. For each $n < \omega$,

$$\mathsf{ZF}+(V = L)+(\mathcal{K} \text{ is } \Pi^1_{N+1}\text{-indescribable}) \vdash \mathcal{K} \in Mh^{\omega_n(I+1)}_{N,n}[\emptyset].$$

2. For any Σ^1_{N+2}-sentences φ, if

$$\mathsf{ZF} + (V = L) + (\mathcal{K} \text{ is } \Pi^1_{n+1}\text{-indescribable}) \vdash \varphi^{L_\mathcal{K}},$$

then we can find an $n < \omega$ such that

$$\mathsf{ZF} + (V = L) + (\mathcal{K} \in Mh^{\omega_n(I+1)}_{N,n}[\emptyset]) \vdash \varphi^{L_\mathcal{K}}.$$

Our proof of Theorem 2.8 is build on [A∞1] with Corollary 1.10 and ordinal analysis in [Rathjen94].

Over $\mathsf{ZF} + (V = L)$ with $\mathcal{K} \in M_N$, the Π^1_{N+1}-indescribability of $\mathcal{K}$ is codified using the L-least counter example $S \in \mathrm{Hull}_{\Sigma_1}(\{\mathcal{K}, \mathcal{K}^+\})$ to the Π^1_{N+1}-indescribability of $\mathcal{K}$.

$$\frac{\vdash \Gamma, \neg\tau^N(S, \mathcal{K}) \quad \vdash \Gamma, \forall \rho \in M_N \cap \mathcal{K}[\tau^N(S, \rho)]}{\vdash \Gamma} \ (\mathbf{Ref}_{\mathcal{K}})$$

where $\tau^N(S, \rho)$ says that S is N-thin(non-stationary)

$$\tau^N(S, \rho) :\Leftrightarrow \exists C \subset \rho[(C \text{ is } N\text{-club})^\rho \wedge (S \cap C = \emptyset)]$$

Proposition 2.9 Let A be a Π^1_{N+1}-sentence, and $\pi \in M_N(X)$.

If $\forall \lambda \in X \cap \pi[L_\lambda \models A]$, then $L_\pi \models A$.

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