An interactive semantics for classical proofs

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Introduction

General motivations

- Model theory
- Recursion theory
- Lambda calculus
- Set theory
- Lattice theory
- Domain theory
- ...

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- **Proof theory**

General motivations

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- **Proof theory**

We need a good **theory of proofs.**

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- Usual soundness and completeness theorems in logic state that

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- The aim of this talk is to show soundness and completeness theorems for **proofs**: roughly speaking,

  π is a proof of **F** if and only if **********.

- I will use tools originally developed for the analysis of **linear logic proofs** in a different context.
- More specifically, the main inspiration is Girard's ludics: ********** is a property determined by **interaction**.

Logic

- Logic = **classical logic**.
- Language = **infinitary formulas**.
- Proof–system = (a variant of) **Tait's calculus**.

Why this kind of logic?

- A *purely logical* approach to (first order, classical) arithmetic.
- All the relevant results also hold for the finitary restriction.

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Why this kind of logic?

- A *purely logical* approach to (first order, classical) arithmetic.
- All the relevant results also hold for the finitary restriction.
- The delicate point is . . . **Contraction** rule.

Contraction

Different "**degrees**" of **contraction**:

- Implicit contraction

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B \vee C} \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B \qquad \vdash \Gamma, C}{\vdash \Gamma, A \wedge B \wedge C}$$

"No" contraction — "No" contraction

$$\frac{\vdash \Gamma, B \vee C, A}{\vdash \Gamma, A \vee B \vee C} \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B \qquad \vdash \Gamma, C}{\vdash \Gamma, A \wedge B \wedge C}$$

Backtracking — Backtracking

$$\frac{\vdash \Gamma, A \vee B \vee C, A}{\vdash \Gamma, A \vee B \vee C} \qquad \frac{\vdash \Gamma, A \wedge B \wedge C, A \qquad \vdash \Gamma, A \wedge B \wedge C, B \qquad \vdash \Gamma, A \wedge B \wedge C, C}{\vdash \Gamma, A \wedge B \wedge C}$$

Full contraction — Full contraction

Main system

- **Formulas**: $\mathbf{F}, \mathbf{G}, \mathbf{H}, \ldots$ generated in the usual way, using connectives $\vee, \wedge, ^{\perp} \ldots$.
- **Sequents** : $\boldsymbol{\Theta}, \boldsymbol{\Phi}, \ldots$ = finite non–empty sequences of formulas $\vdash \mathbf{F}_0, \ldots, \mathbf{F}_{n-1}$.
- **Rules** for deriving sequents.

$$\frac{\{\boldsymbol{\Theta}_a\}_{a \in S}}{\boldsymbol{\Theta}}\ (\mathrm{r})$$

- **Derivations** = well–founded trees labeled by sequent (which are "locally correct").

$$\text{System } \mathcal{A} \stackrel{\text{DEF}}{=} (\mathbf{F}, \mathbf{S}, \mathbf{R}, \mathbf{D})$$

Auxiliary system

- **Formulas**: as in $\mathcal{A}$;
- **Sequents ’** : $\mathbf{\Theta}, \mathbf{\Phi}, \ldots$ = unary sequences of formulas $\vdash_* \mathbf{F}$.
- **Rules ’** for deriving sequents.

$$\frac{\{\mathbf{\Theta}_a\}_{a \in S}}{\mathbf{\Theta}}\ (\mathrm{r})$$

- **Derivations ’** = well–founded trees labeled by sequent (which are “locally correct”).

$$\text{System } \mathcal{B} \stackrel{\text{DEF}}{=} (\mathbf{F}, \mathbf{S'}, \mathbf{R'}, \mathbf{D'})$$

- Every sequent of $\mathcal{B}$ is derivable.

Interaction (I)

- **Cut–elimination** = an operation from trees labeled by sequents to trees labeled by sequents.
- **Closed cuts** = cuts of the form

$$\dfrac{\begin{array}{c}\vdots\,\pi\\ \vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1}\end{array} \qquad \begin{array}{c}\vdots\,\pi_0\\ \vdash_* \mathbf{F}_0^\perp\end{array} \quad \ldots \quad \begin{array}{c}\vdots\,\pi_{n-1}\\ \vdash_* \mathbf{F}_{n-1}^\perp\end{array}}{}\ \textbf{cut}$$

  where π is a derivation of $\vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1}$ in $\mathcal{A}$, and π_i is a derivation of $\vdash_* \mathbf{F}_i^\perp$ in $\mathcal{B}$, for each $i < n$.
- Cut elimination of closed cuts does not produce any cut–free sequent . . .

Interaction (II)

- …but the **procedure** of cut–elimination still makes sense:

$$\dfrac{\dfrac{\dfrac{\vdots\ \pi}{\vdash \mathbf{F} \vee \mathbf{G}, \mathbf{F}}}{\vdash \mathbf{F} \vee \mathbf{G}} \qquad \dfrac{\dfrac{\vdots\ \pi_0}{\vdash_* \mathbf{F}^\perp} \quad \dfrac{\vdots\ \pi_1}{\vdash_* \mathbf{G}^\perp}}{\vdash_* \mathbf{F}^\perp \wedge \mathbf{G}^\perp}}{}\ \textbf{cut}$$

reduces to

$$\dfrac{\dfrac{\vdots\ \pi}{\vdash \mathbf{F} \vee \mathbf{G}, \mathbf{F}} \qquad \dfrac{\dfrac{\vdots\ \pi_0}{\vdash_* \mathbf{F}^\perp} \quad \dfrac{\vdots\ \pi_1}{\vdash_* \mathbf{G}^\perp}}{\vdash_* \mathbf{F}^\perp \wedge \mathbf{G}^\perp} \qquad \dfrac{\vdots\ \pi_0}{\vdash_* \mathbf{F}^\perp}}{}\ \textbf{cut}$$

Interaction (II)

- …but the **procedure** of cut–elimination still makes sense:

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\;\pi\\ \vdash \mathbf{F}\vee\mathbf{G},\mathbf{F}\end{array}}{\vdash \mathbf{F}\vee\mathbf{G}} \qquad \dfrac{\begin{array}{c}\vdots\;\pi_0\\ \vdash_* \mathbf{F}^\perp\end{array} \qquad \begin{array}{c}\vdots\;\pi_1\\ \vdash_* \mathbf{G}^\perp\end{array}}{\vdash_* \mathbf{F}^\perp \wedge \mathbf{G}^\perp}}{}\ \mathbf{cut}
$$

  reduces to

$$
\dfrac{\begin{array}{c}\vdots\;\pi\\ \vdash \mathbf{F}\vee\mathbf{G},\mathbf{F}\end{array} \qquad \dfrac{\begin{array}{c}\vdots\;\pi_0\\ \vdash_* \mathbf{F}^\perp\end{array} \qquad \begin{array}{c}\vdots\;\pi_1\\ \vdash_* \mathbf{G}^\perp\end{array}}{\vdash_* \mathbf{F}^\perp \wedge \mathbf{G}^\perp} \qquad \begin{array}{c}\vdots\;\pi_0\\ \vdash_* \mathbf{F}^\perp\end{array}}{}\ \mathbf{cut}
$$

- We can study the **properties of this procedure.**

Generalization (I)

- We can also consider a more **general version** of closed cuts

$$\dfrac{\dfrac{\vdots\,\pi}{\vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1}} \qquad \dfrac{\vdots\,\pi_0}{\vdash_* \mathbf{G}_0} \quad \ldots \quad \dfrac{\vdots\,\pi_{n-1}}{\vdash_* \mathbf{G}_{n-1}}}{} \ \mathbf{cut}$$

where π is a derivation of $\vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1}$ in $\mathcal{A}$ and π_i is a derivation of $\vdash_* \mathbf{G}_i$ in $\mathcal{B}$, for each $i < n$.

There are new situations to consider:

- **Error**:

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash \mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{F}_1}}{\vdash \mathbf{F}_1 \vee \mathbf{F}_2} \qquad \dfrac{\vdots\,\pi'}{\vdash_* \mathbf{G}_1 \vee \mathbf{G}_2}}{} \ \mathbf{cut}$$

reduces to an "**error**."

Generalization (II)

- **Reduction**:

$$\dfrac{\dfrac{\dfrac{\vdots\ \pi}{\vdash \mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{F}_1}}{\vdash \mathbf{F}_1 \vee \mathbf{F}_2} \qquad \dfrac{\dfrac{\vdots\ \pi_1}{\vdash_* \mathbf{G}_1} \quad \dfrac{\vdots\ \pi_2}{\vdash_* \mathbf{G}_2} \quad \dfrac{\vdots\ \pi_3}{\vdash_* \mathbf{G}_3}}{\vdash_* \mathbf{G}_1 \wedge \mathbf{G}_2 \wedge \mathbf{G}_3}}{} \ \textbf{cut}$$

reduces to

$$\dfrac{\dfrac{\vdots\ \pi}{\vdash \mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{F}_1} \qquad \dfrac{\dfrac{\vdots\ \pi_1}{\vdash_* \mathbf{G}_1} \quad \dfrac{\vdots\ \pi_2}{\vdash_* \mathbf{G}_2} \quad \dfrac{\vdots\ \pi_3}{\vdash_* \mathbf{G}_3}}{\vdash_* \mathbf{G}_1 \wedge \mathbf{G}_2 \wedge \mathbf{G}_3} \qquad \dfrac{\vdots\ \pi_1}{\vdash_* \mathbf{G}_1}}{} \ \textbf{cut}$$

Generalization (II)

- **Reduction**:

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash \mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{F}_1}}{\vdash \mathbf{F}_1 \vee \mathbf{F}_2} \qquad \dfrac{\dfrac{\vdots\,\pi_1}{\vdash_* \mathbf{G}_1} \quad \dfrac{\vdots\,\pi_2}{\vdash_* \mathbf{G}_2} \quad \dfrac{\vdots\,\pi_3}{\vdash_* \mathbf{G}_3}}{\vdash_* \mathbf{G}_1 \wedge \mathbf{G}_2 \wedge \mathbf{G}_3}}{} \ \mathbf{cut}$$

reduces to

$$\dfrac{\dfrac{\vdots\,\pi}{\vdash \mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{F}_1} \qquad \dfrac{\dfrac{\vdots\,\pi_1}{\vdash_* \mathbf{G}_1} \quad \dfrac{\vdots\,\pi_2}{\vdash_* \mathbf{G}_2} \quad \dfrac{\vdots\,\pi_3}{\vdash_* \mathbf{G}_3}}{\vdash_* \mathbf{G}_1 \wedge \mathbf{G}_2 \wedge \mathbf{G}_3} \qquad \dfrac{\vdots\,\pi_1}{\vdash_* \mathbf{G}_1}}{} \ \mathbf{cut}$$

- We can study the **properties of this procedure.**

Generalization (+)

- Instead of considering **derivations** in $\mathcal{A}$, we shall consider **proof–terms**, that we call **tests** $\mathcal{T}, \mathcal{U}, \mathcal{V}, \ldots$
- Intuition:

$$\begin{array}{rcl} \text{Tests} & : & \text{derivations in } \mathcal{A} \\ & = & \\ \text{Untyped lambda terms} & : & \text{derivations in minimal logic} \\ & & \text{(natural deduction)} \end{array}$$

- A test does **not contain all** the information of a derivation. But we can consider closed cuts of the form

$$\dfrac{\mathcal{T} \qquad \vdash_* \mathbf{G}_0 \quad \ldots \quad \vdash_* \mathbf{G}_{n-1}}{}\ \textbf{cut}$$

and define a procedure of reduction (**interaction**).

Trees

Notation

- $\mathbb{N}^* = \{s, t, u, \ldots\}$ = the set of **finite sequences of natural numbers.**
- Some sequences:

$$
\begin{aligned}
(\,) &= \text{the \textbf{empty sequence}};\\
a &= \text{unary sequence};\\
a_0 a_1 &= \text{binary sequence};\\
a_0 a_1 \cdots a_{k-1} &= k\text{–ary sequence.}
\end{aligned}
$$

- st = the **concatenation** of s and t.
- In particular, if s is a k–ary sequence and $a \in \mathbb{N}$, then sa is $(k+1)$–ary sequence.
- **Prefix order**: $s \sqsubseteq t \overset{\text{DEF}}{\iff}$ there is $u \in \mathbb{N}^*$ such that $t = su$.

Trees

- A **tree** T is a non–empty subset of $\mathbb{N}^*$ such that
$$\text{if } t \in T \text{ and } s \sqsubseteq t, \text{ then } s \in T.$$
- Since T is non–empty, $\langle\,\rangle \in T$. $\langle\,\rangle$ is called the **root** of T.
- An **infinite branch** in T is a infinite subset $S \subseteq T$ of the form $S = \{\langle\,\rangle\,,\ a_0\,,\ a_0 a_1\,,\ \ldots\,,\ a_0 a_1 \cdots a_{n-1}\,,\ \ldots\}$.
- A tree is said to be **well–founded** if it does not contain an infinite branch.
- A **labeled tree** is a pair $L = (T, \varphi)$ consisting of a tree T and a function φ defined on T.
- φ is called the **labeling function** of L. The codomain of φ is called the set of **labels**.
- We write $\operatorname{tree}(L)$ and $\operatorname{lab}(L)$ for the underlying tree of L and its labeling function respectively, i.e., if $L = (T, \varphi)$, then $\operatorname{tree}(L) = T$ and $\operatorname{lab}(L) = \varphi$.

System $\mathcal{A}$

System $\mathcal{A}$

System $\mathcal{A}$ is a variant of **Tait's calculus** (1968).

- Finite sequences instead of finite sets.
- No propositional variables in this talk.
- Only subsets of natural numbers as index sets.

Formulas

The **formulas** of our language are inductively defined as follows:

> *if for some $S \subseteq \mathbb{N}$, $\{\mathbf{G}_a\}_{a \in S}$ is a family of formulas, then $\bigvee_S \mathbf{G}_a$ and $\bigwedge_S \mathbf{G}_a$ are formulas.*

Some terminology and notation:

- $\bigvee_S \mathbf{G}_a$ = **disjunction**;
- $\bigwedge_S \mathbf{G}_a$ = **conjunction**;
- $\mathbf{0} \stackrel{\text{DEF}}{=} \bigvee_{\emptyset} \mathbf{G}_a$;
- $\mathbf{1} \stackrel{\text{DEF}}{=} \bigwedge_{\emptyset} \mathbf{G}_a$.

Negation and sequents

The **negation** of a formula $\mathbf{F}$, noted by $\mathbf{F}^\perp$, is the formula recursively defined as follows:

$$(\bigvee_S \mathbf{G}_a)^\perp \stackrel{\text{DEF}}{=} \bigwedge_S (\mathbf{G}_a{}^\perp); \qquad (\bigwedge_S \mathbf{G}_a)^\perp \stackrel{\text{DEF}}{=} \bigvee_S (\mathbf{G}_a{}^\perp).$$

In particular, $\mathbf{0}^\perp = \mathbf{1}$, and $\mathbf{1}^\perp = \mathbf{0}$.

The negation is **involutive**:

$$\mathbf{F}^{\perp\perp} = \mathbf{F}.$$

A **sequent** $\boldsymbol{\Theta}, \boldsymbol{\Phi}, \ldots$ of $\mathcal{A}$ is a non–empty finite sequence $\vdash \ \mathbf{F}_0, \ldots, \mathbf{F}_{n-1}$ of formulas ($n > 0$).

Rules

The following **rules** derive *sequents*. They have to be read bottom–up, in the sense of *proof–search*.

Disjunctive rule :

- $i < n$ and $a_0 \in S$:

$$\frac{\vdash\ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1}\ ,\ \bigvee_S \mathbf{G}_a\ ,\ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}\ ,\ \mathbf{G}_{a_0}}{\vdash\ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1}\ ,\ \bigvee_S \mathbf{G}_a\ ,\ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}}\ (\vee)$$

Conjunctive rule :

- $i < n$, one premise for each member of S:

$$\frac{\vdash\ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1}\ ,\ \bigwedge_S \mathbf{G}_a\ ,\ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}\ ,\ \mathbf{G}_a \qquad \ldots \text{all } a \in S}{\vdash\ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1}\ ,\ \bigwedge_S \mathbf{G}_a\ ,\ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}}\ (\wedge)$$

Derivations

A **derivation** is a **well–founded tree labeled by sequents** which is "locally correct." Formally,

A derivation is a well–founded tree π labeled by sequents such that for all $s \in \mathrm{tree}(\pi)$ one of the following two conditions holds:

$$
\textbf{(Der}_1\textbf{)}:\left\{\begin{array}{ll}
\text{(i)} & \mathrm{lab}(\pi)(s) \text{ is a sequent } \vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1} \text{ and} \\
& \text{there are } i<n \text{ and } a_0\in\mathbb{N} \text{ such that} \\
& \mathbf{F}_i = \bigvee_S \mathbf{G}_a \text{ and } a_0 \in S, \\
\text{(ii)} & sa \in \mathrm{tree}(\pi) \text{ if and only if } a=0, \\
\text{(iii)} & \mathrm{lab}(\pi)(s0) = \vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1},\mathbf{G}_{a_0}.
\end{array}\right.
$$

$$
\textbf{(Der}_2\textbf{)}:\left\{\begin{array}{ll}
\text{(i)} & \mathrm{lab}(\pi)(s) \text{ is a sequent } \vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1} \text{ and} \\
& \text{there is } i<n \text{ such that } \mathbf{F}_i = \bigwedge_S \mathbf{G}_a, \\
\text{(ii)} & sa \in \mathrm{tree}(\pi) \text{ if and only if } a\in S, \\
\text{(iii)} & \mathrm{lab}(\pi)(sa) = \vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1},\mathbf{G}_a, \text{ for all } a\in S.
\end{array}\right.
$$

Some derivable sequents

- A derivation with no premises is

$$\frac{}{\vdash \ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \mathbf{1} \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}} \ (\wedge)$$

  - Every leaf of a derivation is labeled by a sequent of this form.
- Sequents of this form are derivable:

$$\vdash \ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \mathbf{G} \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{j-1} \ , \ \mathbf{G}^{\perp} \ , \ \mathbf{F}_{j+1}, \ldots, \mathbf{F}_{n-1}$$

- **Novikoff's law of complete induction** is the formula

$$\big(F_1 \wedge (F_1 \to F_2) \wedge (F_2 \to F_3) \wedge \cdots\big) \to F_1 \wedge F_2 \wedge F_3 \wedge \cdots .$$

  In our system, we can consider the sequent

$$\vdash \ \big(\mathbf{F}_1^{\perp} \vee (\mathbf{F}_1 \wedge \mathbf{F}_2^{\perp}) \vee (\mathbf{F}_2 \wedge \mathbf{F}_3^{\perp}) \vee \cdots\big) \ , \ \mathbf{F}_1 \wedge \mathbf{F}_2 \wedge \mathbf{F}_3 \wedge \cdots .$$

  and show that it is derivable.

Tests

Actions

- A **disjunctive action** is a triple $\langle n, i, a \rangle$ where n, i, a are natural numbers such that $0 \leq i < n$.
- A **conjunctive action** is a triple $[n, i, S]$ where n, i are natural numbers such that $0 \leq i < n$, and $S \subseteq \mathbb{N}$.

Some terminology:

- $\langle n, i, a \rangle = \langle$ **base** , **address** , **name** $\rangle$;
- $[n, i, S] = [$ **base** , **address** , **set of names** $]$;

Tests

A ***test*** *is a tree* $\mathcal{T}$ *labeled by actions such that for all* $s \in \mathrm{tree}(\mathcal{T})$ *one of the following two conditions holds:*

$$\textbf{(T}_1\textbf{)} : \begin{cases} \text{(i)} & \mathrm{lab}(\mathcal{T})(s) = \langle n, i, a_0 \rangle, \\ \text{(ii)} & sa \in \mathrm{tree}(\mathcal{T}) \text{ if and only if } a = 0, \\ \text{(iii)} & \text{the base of } \mathrm{lab}(\mathcal{T})(s0) \text{ is } n+1. \end{cases}$$

$$\textbf{(T}_2\textbf{)} : \begin{cases} \text{(i)} & \mathrm{lab}(\mathcal{T})(s) = [n, i, S], \\ \text{(ii)} & sa \in \mathrm{tree}(\mathcal{T}) \text{ if and only if } a \in S, \\ \text{(iii)} & \text{the base of } \mathrm{lab}(\mathcal{T})(sa) \text{ is } n+1, \\ & \text{for all } a \in S. \end{cases}$$

We use letters $\mathcal{T}, \mathcal{U}, \mathcal{V}, \ldots$ to range over tests.

- Tests are not necessarily well–founded.

Terminology and notation

Let $\mathcal{T}$ be a test.

- If the action $\mathrm{lab}(\mathcal{T})(())$ has base n, we say that $\mathcal{T}$ **is on base** n.
- If $\mathrm{lab}(\mathcal{T})(()) = \langle n, i, a_0 \rangle$, then $\mathcal{T}$ has a unique immediate subtree $\mathcal{U}$. We denote $\mathcal{T}$ by $\langle n, i, a_0 \rangle.\mathcal{U}$.
- If $\mathrm{lab}(\mathcal{T})(()) = [n, i, S]$, then $\mathcal{T}$ has an immediate subtree $\mathcal{U}_a$ for each $a \in S$. We denote $\mathcal{T}$ by $[n, i, S].\mathcal{U}_a$.
  If $S = \emptyset$, then we simply write $[n, i, \emptyset]$.

$\mathcal{T} \rhd \mathbf{\Theta}$

Let π be a derivation of $\mathbf{\Theta}$ in $\mathcal{A}$.
We define the relation $\mathcal{T} \rhd \mathbf{\Theta}$ between tests and sequents of $\mathcal{A}$ inductively as follows:

$$\frac{\mathcal{U} \rhd \vdash \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \bigvee_S \mathbf{G}_a \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1} \ , \ \mathbf{G}_{a_0}}{\langle n, i, a_0 \rangle . \mathcal{U} \rhd \vdash \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \bigvee_S \mathbf{G}_a \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}} \ (\vee)$$

$$\frac{\mathcal{U}_a \rhd \vdash \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} , \bigwedge_S \mathbf{G}_a , \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}, \mathbf{G}_a \ \ldots \text{all } a \in S}{[n, i, S] . \mathcal{U}_a \rhd \vdash \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} , \bigwedge_S \mathbf{G}_a , \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}} \ (\wedge)$$

Properties of $\mathcal{T} \rhd \Theta$

- **Bijective correspondence** between

  $\{\mathcal{T} : \mathcal{T} \rhd \Theta\}$ and $\{\pi : \pi$ is a derivation of Θ in $\mathcal{A}\}$.
- If $\mathcal{T} \rhd \Theta$, then $\mathcal{T}$ is **well–founded.**
- The relation $\mathcal{T} \rhd \Theta$ is defined **syntactically**, i.e., using derivations.
- Later on, we shall define a relation $\mathcal{T} \blacktriangleright \Theta$ **interactively**, i.e., using a kind of cut–elimination procedure.

Counter–tests

System $\mathcal{B}$

We now consider another proof–system, that we call **system** $\mathcal{B}$:

- **Formulas** : as in $\mathcal{A}$
- **Sequents '** : A sequent of $\mathcal{B}$ is a unary sequence of formulas $\vdash_* \mathbf{F}$.
- **Rules '** :
  - **Disjunctive rule**: one premise for each $a \in S$:

$$\frac{\vdash \ \mathbf{G}_a \quad \ldots \text{all } a \in S}{\vdash \ \bigvee_S \mathbf{G}_a} \ (\vee')$$

  - **Conjunctive rule**: one premise for each $a \in S$:

$$\frac{\vdash \ \mathbf{G}_a \quad \ldots \text{all } a \in S}{\vdash \ \bigwedge_S \mathbf{G}_a} \ (\wedge')$$

- **Derivations '** : well–founded trees labeled by sequents of $\mathcal{B}$ which are "locally correct."

Remarks and terminology

- For every formula **F** there is one (and only one) derivation of $\vdash_* \mathbf{F}$ in $\mathcal{B}$. By an abuse of notation we write $\vdash_* \mathbf{F}$ for the derivation of this sequent in $\mathcal{B}$.
- For any formula **F**, we call **the** derivation of $\vdash_* \mathbf{F}$ in $\mathcal{B}$ a **counter–test**.
- A derivation of $\vdash_* \mathbf{F}$ in $\mathcal{B}$ can be seen as the **subformula tree (in the sense of Gentzen)** of **F**.
- **For the formulas we are considering**,

  subformula a'la Gentzen = literal subformula.

Interaction, soundness and completeness

Configurations

A **configuration** is either

- a pair $(\mathcal{T}, \vdash_* \mathbf{G}_0, \ldots, \vdash_* \mathbf{G}_{n-1})$ where:
  - $\mathcal{T}$ is a **test** of base n,
  - $\vdash_* \mathbf{G}_0, \ldots, \vdash_* \mathbf{G}_{n-1}$ is a n–ary sequence of counter–tests,

  for some $n > 0$;
- or the symbol $\Uparrow$ (**error**).

$\mathbb{C}$ denotes the set of all configurations.

- Intuition:

$$(\mathcal{T}, \vdash_* \mathbf{G}_0, \ldots, \vdash_* \mathbf{G}_{n-1}) \quad \approx \quad \frac{\vdash \mathbf{F}_0, \ldots, \mathbf{F}_{n-1} \qquad \vdash_* \mathbf{G}_0 \quad \ldots \quad \vdash_* \mathbf{G}_{n-1}}{}\,\textbf{cut}$$

Reduction relation (I)

The **reduction relation** $\longrightarrow$ is the subset of $\mathbb{C} \times \mathbb{C}$ defined as follows.

(1) $\Uparrow \longrightarrow \Uparrow$.

- Intuition: “ error reduces to error.”

Reduction relation (II)

(2) Let $C = (\langle n, i, a_0\rangle.\mathcal{U}\ ,\ \vdash_* \mathbf{G}_0\ \ \ldots\ \ \vdash_* \mathbf{G}_{n-1})$.

- If $\mathbf{G}_i = \bigwedge_S \mathbf{G}_a$ and $a_0 \in S$, then
$$C \longrightarrow\ (\mathcal{U}\ ,\ \vdash_* \mathbf{G}_0\ \ \ldots\ \ \vdash_* \mathbf{G}_{n-1} \vdash_* \mathbf{G}_{a_0}).$$
- $C\ \longrightarrow \Uparrow$, otherwise.
  - Intuition (case $n = 2$ and $i = 1$):

$$\dfrac{\dfrac{\overset{\vdots\,\pi}{\vdash \mathbf{F}_0\ ,\ \bigvee_T \mathbf{H}_a\ ,\ \mathbf{H}_{a_0}}}{\vdash \mathbf{F}_0\ ,\ \bigvee_T \mathbf{H}_a}\,(\vee) \qquad \overset{\vdots\,\pi_0}{\vdash_* \mathbf{G}_0} \qquad \dfrac{\overset{\vdots\,\pi_a}{\vdash_* \mathbf{G}_a} \quad \ldots \text{all } a \in S}{\vdash_* \bigwedge_S \mathbf{G}_a}\,(\wedge')}{}\ \textbf{cut}$$

reduces to

$$\dfrac{\overset{\vdots\,\pi}{\vdash \mathbf{F}_0\ ,\ \bigvee_T \mathbf{H}_a\ ,\ \mathbf{H}_{a_0}} \quad \overset{\vdots\,\pi_0}{\vdash_* \mathbf{G}_0} \quad \dfrac{\overset{\vdots\,\pi_a}{\vdash_* \mathbf{G}_a} \ldots \text{all } a \in S}{\vdash_* \bigwedge_S \mathbf{G}_a}\,(\wedge') \quad \overset{\vdots\,\pi_{a_0}}{\vdash_* \mathbf{G}_{a_0}}}{}\ \textbf{cut}$$

Reduction relation (III)

(3) Let $C = \big([n, i, T].\mathcal{U}_a\ ,\ \vdash_* \mathbf{G}_0\ \ \ldots\ \ \vdash_* \mathbf{G}_{n-1}\big)$.

- If $\mathbf{G}_i = \bigvee_S \mathbf{G}_a$ and $S = T$, then

$$C \longrightarrow \ \big(\mathcal{U}_a\ ,\ \vdash_* \mathbf{G}_0\ \ \ldots\ \ \vdash_* \mathbf{G}_{n-1} \vdash_* \mathbf{G}_a\big)\ \ ,\ \ \text{for all } a \in S.$$

- $C \ \longrightarrow \Uparrow$, otherwise.

  - Intuition (case $n = 2$ and $i = 1$):

$$\cfrac{\cfrac{\begin{array}{c}\vdots\,\pi\\ \vdash \mathbf{F}_0\,,\ \bigwedge_S \mathbf{H}_a\,,\ \mathbf{H}_{a_0}\end{array}}{\vdash \mathbf{F}_0\,,\ \bigwedge_S \mathbf{H}_a}\,(\wedge) \qquad \begin{array}{c}\vdots\,\pi_0\\ \vdash_* \mathbf{G}_0\end{array} \qquad \cfrac{\begin{array}{c}\vdots\,\pi_a\\ \vdash_* \mathbf{G}_a\end{array}\quad \ldots \text{all } a\in S}{\vdash_* \bigvee_S \mathbf{G}_a}\,(\vee')}{}\ \mathbf{cut}$$

reduces to

$$\cfrac{\begin{array}{c}\vdots\,\pi\\ \vdash \mathbf{F}_0\,,\ \bigwedge_S \mathbf{H}_a\,,\ \mathbf{H}_a\end{array} \quad \begin{array}{c}\vdots\,\pi_0\\ \vdash_* \mathbf{G}_0\end{array} \quad \cfrac{\begin{array}{c}\vdots\,\pi_a\\ \vdash_* \mathbf{G}_a \ldots \text{all } a\in S\end{array}}{\vdash_* \bigvee_S \mathbf{G}_a}\,(\vee') \quad \begin{array}{c}\vdots\,\pi_a\\ \vdash_* \mathbf{G}_a\end{array}}{}\ \mathbf{cut}$$

Some properties of $\longrightarrow$ (I)

Let A be a set and let R be a binary relation of A.

- R is **total** $\overset{\text{DEF}}{\iff}$ for all $a \in A$ there is $b \in A$ such that $a \mathrel{R} b$;
- R is **deterministic** $\overset{\text{DEF}}{\iff}$ $a \mathrel{R} b$ and $a \mathrel{R} c$ imply $b = c$;
- R is **terminating** $\overset{\text{DEF}}{\iff}$ there is no infinite sequence
$$a_0 \longrightarrow a_1 \longrightarrow \cdots .$$

The relation $\longrightarrow$ is **not total**:

$\big([1, 0, S].\mathcal{U}_a \ , \ \vdash_* \bigvee_S \mathbf{G}_a\big)$ does not reduce to anything, if $S = \emptyset$.

The relation $\longrightarrow$ is **not deterministic**:

$$\big([1, 0, \{c, d\}].\mathcal{U}_a \ , \ \vdash_* \bigvee_{\{c,d\}} \mathbf{G}_a\big) \text{ reduces to}$$
$$\big(\mathcal{U}_c \ , \ \vdash_* \bigvee_{\{c,d\}} \mathbf{G}_a \vdash_* \mathbf{G}_c\big) \quad \text{and} \quad \big(\mathcal{U}_d \ , \ \vdash_* \bigvee_{\{c,d\}} \mathbf{G}_a \vdash_* \mathbf{G}_d\big)$$

Some properties of $\longrightarrow$ (II)

The relation $\longrightarrow$ is **not terminating**:

$$\Uparrow \longrightarrow \Uparrow \longrightarrow \cdots$$

A more interesting example is the following:

- $\mathcal{T} \stackrel{\text{DEF}}{=} \langle 1, 0, a_0\rangle.\langle 2, 0, a_0\rangle \ldots \langle n, 0, a_0\rangle.\langle n+1, 0, a_0\rangle \ldots$;
- $\mathbf{F} \stackrel{\text{DEF}}{=} \bigwedge_{\{a_0\}} \mathbf{G}_a$, where $\mathbf{G}_{a_0} \stackrel{\text{DEF}}{=} \mathbf{0}$.

$$\begin{aligned}
(\mathcal{T}\ , \vdash_* \mathbf{F}) &\longrightarrow (\langle 2, 0, a_0\rangle \ldots\ , \vdash_* \mathbf{F} \vdash_* \mathbf{0}) \\
&\longrightarrow \\
&\ \vdots \\
&\longrightarrow (\langle n, 0, a_0\rangle.\langle n+1, 0, a_0\rangle \ldots\ , \vdash_* \mathbf{F} \vdash_* \mathbf{0} \ldots \vdash_* \mathbf{0}) \\
&\longrightarrow (\langle n+1, 0, a_0\rangle \ldots\ , \vdash_* \mathbf{F} \vdash_* \mathbf{0} \ldots \vdash_* \mathbf{0} \vdash_* \mathbf{0}) \\
&\longrightarrow \\
&\ \vdots
\end{aligned}$$

$D \blacktriangleright \Theta$

We now define the relation $\mathcal{T} \blacktriangleright \Theta$, the **semantical** counterpart of the relation $\mathcal{T} \triangleright \Theta$.

> $\mathcal{T} \blacktriangleright \vdash \mathbf{F}_0, \ldots, \mathbf{F}_{n-1} \overset{\text{DEF}}{\Longleftrightarrow}$ *every sequence of reductions starting from* $(\mathcal{T}, \vdash_* \mathbf{F}_0^{\perp} \ldots \vdash_* \mathbf{F}_{n-1}^{\perp})$ *terminates.*

Soundness and completeness :

$$\mathcal{T} \triangleright \Theta \Longleftrightarrow \mathcal{T} \blacktriangleright \Theta.$$

VARIANTS

$\mathcal{T} \ \triangleright' \ \boldsymbol{\Theta}$

Let π be a derivation of $\boldsymbol{\Theta}$ in $\mathcal{A}$. The relation $\mathcal{T} \ \triangleright' \ \boldsymbol{\Theta}$ is defined inductively as follows:

$$\frac{\mathcal{U} \ \triangleright' \ \vdash \ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \bigvee_S \mathbf{G}_a \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1} \ , \ \mathbf{G}_{a_0}}{\langle n, i, a_0 \rangle . \mathcal{U} \ \triangleright' \ \vdash \ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \bigvee_S \mathbf{G}_a \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}} \ (\vee)$$

$$\frac{\mathcal{U}_a \triangleright' \ \vdash \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \bigwedge_S \mathbf{G}_a \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}, \ \mathbf{G}_a \ldots \text{all } a \in S}{[n, i, T] . \mathcal{U}_a \triangleright' \ \vdash \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \bigwedge_S \mathbf{G}_a \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}} \ (\wedge)$$

where $S \subseteq T$ and $\mathcal{U}_b$ is an arbitrary test, for each $b \in T \setminus S$.

- If $\mathcal{T} \ \triangleright' \ \boldsymbol{\Theta}$, then $\mathcal{T}$ is **not necessarily well–founded.**
- $\{\mathcal{T} : \mathcal{T} \triangleright \ \boldsymbol{\Theta}\} \subsetneq \{\mathcal{T} : \mathcal{T} \triangleright' \ \boldsymbol{\Theta}\}$.

Reduction relation $\longrightarrow'$

The **reduction relation** $\longrightarrow'$ is the subset of $\mathbb{C} \times \mathbb{C}$ defined as follows.

(1) $\Uparrow \longrightarrow' \Uparrow$.

(2) Let $C = (\langle n, i, a_0 \rangle.\mathcal{U} \ , \vdash_* \mathbf{G}_0 \ \ \dots \ \ \vdash_* \mathbf{G}_{n-1})$.

- If $\mathbf{G}_i = \bigwedge_S \mathbf{G}_a$ and $a_0 \in S$, then
$$C \longrightarrow' \ (\mathcal{U} \ , \vdash_* \mathbf{G}_0 \ \ \dots \ \ \vdash_* \mathbf{G}_{n-1} \vdash_* \mathbf{G}_{a_0}).$$
- $C \ \longrightarrow' \Uparrow$, otherwise.

(3) Let $C = ([n, i, T].\mathcal{U}_a \ , \vdash_* \mathbf{G}_0 \ \ \dots \ \ \vdash_* \mathbf{G}_{n-1})$.

- If $\mathbf{G}_i = \bigvee_S \mathbf{G}_a$ and $S \subseteq T$, then
$$C \longrightarrow' \ (\mathcal{U}_a \ , \vdash_* \mathbf{G}_0 \ \ \dots \ \ \vdash_* \mathbf{G}_{n-1} \vdash_* \mathbf{G}_a) \ \ , \ \text{for all } a \in S.$$
- $C \ \longrightarrow' \Uparrow$, otherwise.

$D \blacktriangleright' \Theta$

We now define the relation $\mathcal{T} \blacktriangleright' \Theta$, the **semantical** counterpart of the relation $\mathcal{T} \triangleright' \Theta$.

$\mathcal{T} \blacktriangleright' \vdash \mathbf{F}_0, \ldots, \mathbf{F}_{n-1} \overset{\text{DEF}}{\Longleftrightarrow}$ *every sequence of* $\longrightarrow'$ *reductions starting from* $(\mathcal{T}, \vdash_* \mathbf{F}_0^\perp \ldots \vdash_* \mathbf{F}_{n-1}^\perp)$ *terminates.*

Soundness and completeness :

$$\mathcal{T} \triangleright' \Theta \iff \mathcal{T} \blacktriangleright' \Theta.$$

FURTHER WORK

Further work

- Propositional variables and **second order** quantifiers.
- Girard's β–logic (the logic underlying the theory of **dilators**).
- ...

Thank you!

Thank you!

Questions?

Thank you!

Questions?

Answers?