Inequality

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Consider the following axioms.

(A) $\quad x \neq y \Rightarrow x \neq z \vee y \neq z$

(B)[1] $\quad x \leq y \wedge x \neq y \Rightarrow x \leq z \vee z \leq y$

(C) $\quad x \neq y \Rightarrow x \leq y \vee y \leq x$

[1] This axiom was suggested by Douglas S. Bridges.

Minimalistic setting

Let $<$ a binary relation on a set X such that

- $\neg(x < x)$ (irreflexive)
- $x < y \wedge y < z \Rightarrow x < z$ (transitive)
- $x < y \Rightarrow x < z \vee z < y$ (approximate splitting)

Set

$$x \leq y \overset{def}{\Leftrightarrow} \neg(y < x)$$

$$x = y \overset{def}{\Leftrightarrow} x \leq y \wedge y \leq x$$

$$x \neq y \overset{def}{\Leftrightarrow} \neg(x = y)$$

Minimalistic setting

With classical logic, (A), (B), and (C) are true.

What can be said with intuitionistic logic?

Minimalistic setting, (A) $\wedge$ (C) $\Rightarrow$ (B)

Fix x, y, z and assume that $x \neq y$ and $x \leq y$.

By (A) we either have $x \neq z$ or $y \neq z$.

Considering the first case, (C) gives us either $x \leq z$, which is fine, or $z \leq x$, which implies $z \leq y$.

The second case is treated analogously.

Group setting

Suppose that there exist an element 0 of X, and a functions $+$, max from $X \times X$ into X such that

- $(X, +, 0)$ is an Abelian group
- $x < y \Rightarrow x + z < y + z$
- $0 \leq \max(x, -x)$
- $x < y \;\Rightarrow\; max(x, y) = max(y, x) = y$

Proposition

$(A) \iff (B) \implies (C)$

Group setting, $(A) \Rightarrow (C)$

Fix x, y and assume that $x \neq y$.

Set $z = \max(x, y)$.

By (A) we have either $x \neq z$ or $y \neq z$.

Suppose that $x \neq z$.

Then $x \leq y$, because $y < x$ would imply $x = z$

The case $y \neq z$ is treated analogously.

This implies $(A) \Rightarrow (B)$ as well.

Group setting, $(B) \Rightarrow (A)$

Fix x, y with $x \neq y$. We show that either $x \neq 0$ or $y \neq 0$. Set

$a = -\max(x, -x)$

$b = \max(y, -y)$

$c = a + a + b + b$

We have $a \leq b$ and $a \neq b$.

If $a + a + b + b \leq b$, then $b \leq -a - a$ and therefore $x \neq 0$.

If $a \leq a + a + b + b$, then $-a \leq b + b$ and therefore $y \neq 0$.

Real number setting

The set $\mathbb{R}$ of the **Cauchy reals** $\mathbb{R}$ is the set of all rational sequences $x = (x_n)$ such that

$$\forall m, n \left(|x_m - x_n| \leq 2^{-m} + 2^{-n} \right).$$

For two reals x, y we define

$$x < y \overset{def}{\Leftrightarrow} \exists n \left(x_n + 2^{-n+1} < y_n \right).$$

Real number setting

Proposition

$(A) \Leftrightarrow (B) \Leftrightarrow (C) \Leftrightarrow \Pi^0_1\text{-}\mathrm{DML}$

Where $\Pi^0_1\text{-}\mathrm{DML}$ says that

$$\neg(\Phi \wedge \Psi) \Rightarrow \neg\Phi \vee \neg\Psi$$

for Π^0_1-formulas Φ and Ψ.[2]

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[2] A formula Φ is a Π^0_1-*formula* if there exists a binary sequence α such that

$$\Phi \leftrightarrow \forall n\,(\alpha n = 0)\,.$$

Real number setting

The proof of $\Pi^0_1\text{-}\mathrm{DML} \Rightarrow (A)$ is simple.

We show $(C) \Rightarrow \Pi^0_1\text{-}\mathrm{DML}$.

Real number setting, $(C) \Rightarrow \Pi^0_1\text{-DML}$

Fix binary sequences α, β such that

$$\neg\left(\forall n\,(\alpha n = 0) \wedge \forall n\,(\beta n = 0)\right).$$

We have to show that

$$\neg\forall n\,(\alpha n = 0) \vee \neg\forall n\,(\beta n = 0).$$

Define binary sequences α' and β' by

$$\alpha' n = 1 \overset{def}{\Leftrightarrow} \alpha n = 1 \wedge \forall k < n\,(\alpha k = 0 \wedge \beta k = 0)$$

$$\beta' n = 1 \overset{def}{\Leftrightarrow} \beta n = 1 \wedge \forall k < n\,(\alpha k = 0 \wedge \beta k = 0) \wedge \alpha n = 0$$

Real number setting, $(C) \Rightarrow \Pi^0_1$-DML

Define sequences $x = (x_n)$ and $y = (y_n)$ by

$$x_0 = y_0 = 0,$$

and for positive n,

$$x_n = \begin{cases} 2^{-k} & \text{if there exists } k \leq n \text{ with } \alpha' k = 1 \\ 0 & \text{else} \end{cases}$$

$$y_n = \begin{cases} 2^{-k} & \text{if there exists } k \leq n \text{ with } \beta' k = 1 \\ 0 & \text{else} \end{cases}$$

Real number setting, $(C) \Rightarrow \Pi^0_1$-DML

Note that

- x and y are real numbers
- $x = 0 \Leftrightarrow \forall n\,(\alpha' n = 0)$
- $y = 0 \Leftrightarrow \forall n\,(\beta' n = 0)$
- $x = y \Rightarrow x = 0 \wedge y = 0$
- $\neg\,(\forall n\,(\alpha' n = 0) \wedge \forall n\,(\beta' n = 0))$

So x and y are real numbers with $x \neq y$. By (C), we obtain

$$x \leq y \vee y \leq x.$$

The case $x \leq y$ implies $\neg\forall n\,(\beta' n = 0)$, which in turn implies $\neg\forall n\,(\beta n = 0)$. The case $y \leq x$ implies $\neg\forall n\,(\alpha n = 0)$.