The Muchnik degrees of Π^0_1 and Σ^1_1 classes

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Aim

- $\mathcal{P}_{\mathrm{w}}$: the Muchnik degrees of nonempty Π^0_1 subsets of Cantor space 2^ω.
- A long standing simple question:
  $a < b \Longrightarrow (a, b) \cap \mathcal{P}_{\mathrm{w}} \neq \emptyset$?
- Let $\mathcal{S}_{\mathrm{w}}$ be the Muchnik degrees of Σ^1_1 subsets of Baire space ω^ω.
- One of the main theorems of this talk says: $\forall a, b \in \mathcal{P}_{\mathrm{w}}$,
  $[(a, b) \cap \mathcal{S}_{\mathrm{w}} \neq \emptyset \iff (a, b) \cap \mathcal{P}_{\mathrm{w}} \neq \emptyset]$.

Medvedev and Muchnik Degrees

- Medvedev/Muchnik reducibilities $\leq_s$, $\leq_w$ are natural extensions of Turing reducibility $\leq_T$ for subsets of ω^ω.
- def(Medvedev, 1955).
  $P \leq_s Q \iff \exists \text{rec } \Phi \forall f,\ \Phi(f) \in P$.
- def(Muchnik, 1963).
  $P \leq_w Q \iff \forall f \exists \text{rec } \Phi,\ \Phi(f) \in P$.
- For each $r \in \{s, w\}$, $\leq_r$ is a pre-order.
  Thus $P \equiv_r Q \iff P \leq_r Q \leq_r P$ is an equiv. relation.
  We define $\mathcal{D}_r = \mathrm{Pow}(\omega^\omega)/\equiv_r$,
  $\mathcal{P}_r = \{\text{nonempty } \Pi^0_1 \text{ subsets of } 2^\omega\}/\equiv_r$
  and $\mathcal{S}_r = \{\Sigma^1_1 \text{ subsets of } 2^\omega\}/\equiv_r$.

Π^0_1 *Subsets of* 2^ω

- Important examples of Π^0_1 subsets of 2^ω are the sets of all (codes of) completions of some recursive theories of first-order logic.
  - ZFC, PA, RCA, ACA, $Z_2, \cdots$.
- $\mathcal{P}_{\mathrm{s}}$, $\mathcal{P}_{\mathrm{w}}$ have been well studied since around 2000.
  However, the Turing degrees of elements of Π^0_1 subsets of 2^ω was investigated before.
  E.g. Jockusch/Soare (1972) proved each nonempty Π^0_1 contains a low element f, i.e., $f' \leq_{\mathrm{T}} \emptyset'$.
- Π^0_1 is the first level of the arithmetical hierarchies such that the degree structure is non-trivial.
- I do not know any study on $\mathcal{S}_{\mathrm{s}}$ or $\mathcal{S}_{\mathrm{w}}$.
- (Kleene) The set of all non-hyperarithmetic reals is an example of Σ^1_1 subsets of ω^ω.

Known Results

- Cenzer/Hinman, 2003: $\mathcal{P}_s$ is dense.
- Binns, 2003:
  In $\mathcal{P}_s$, $a < b \Longrightarrow \exists c, d,\ a < c, d < b = \sup(c, d)$.
  In $\mathcal{P}_w$, $0 < b \Longrightarrow \exists c, d,\ 0 < c, d < b = \sup(c, d)$.
- Cole/Kihara, 2010: The Σ^0_2-theory of $\mathcal{P}_s$ as an order structure is decidable.
  Here note "$\forall a, b[a < b \Longrightarrow \exists c,\ a < c < b]$" is Π^0_2.
- Shafer, 2012: The Σ^0_4-theory of $\mathcal{P}_s$ and $\mathcal{P}_w$ as order structures is undecidable.
- Is $\mathcal{P}_w$ dense?

Lattice $\mathcal{D}_{\mathrm{w}}$

Recall $\mathcal{D}_{\mathrm{w}} = \mathrm{Pow}(\omega^\omega)/\equiv_{\mathrm{w}}$.
$\mathcal{D}_{\mathrm{w}}$ is a distributive lattice with the top and the bot:

- For $P, Q \subset \omega^\omega$,
  $P \times Q = \{f \oplus g : f \in P, g \in Q\}$ induces the sup,
  where $f \oplus g(2n) = f(n)$ and $f \oplus g(2n+1) = g(n)$.
- $P \cup Q$ induces the inf.
- The deg of P is bottom
  $\iff$ P contains a computable element.
- The deg of Q is top
  $\iff$ $Q = \emptyset$.
  Later, sometimes the condition "$Q <_{\mathrm{w}} \emptyset$" will be appeared. This is equivalent to say "Q is nonempty".

Open Intervals in $\mathcal{D}_{\mathrm{w}}$

- It is not hard to see the following: $\forall P, Q \subset \omega^{\omega}$ in $\mathcal{D}_{\mathrm{w}}$, $(\deg_{\mathrm{w}}(P), \deg_{\mathrm{w}}(Q))$ is empty $\iff (\exists f \in P)[\{f\} \equiv_{\mathrm{w}} P, \{g :>_{\mathrm{T}} f\} \equiv_{\mathrm{w}} Q]$.
- As a corollary, $\mathcal{S}_{\mathrm{w}}$ is not dense.

Lattice $\mathcal{P}_{\rm w}$

Let $\mathcal{P}_{\rm w}$ be the set of all weak degrees of nonempty Π^0_1 subsets of 2^ω.

$\mathcal{P}_{\rm w}$ is a distributive lattice with the top and the bot:

- $P \times Q$ and $P \cup Q$ induce the sup and the inf, resp.
- The deg of P is bottom
  $\iff$ P contains a computable element.
- The deg of PA is top,
  where PA denotes the set of all (codes of) consistent complete extensions of Peano Arithmetic.
- There are some interesting intermediate degrees in $\mathcal{P}_{\rm w}$.
  E.g., $\deg_{\rm w}(\mathrm{MLR}) \in \mathcal{P}_{\rm w}$,
  where MLR is the set of all Martin-Löf random reals.
  (In fact, MLR is a Σ^0_2 subset of 2^ω, though.)

The Embedding Lemma

- The Embedding Lem(Simpson).
  $\forall \Sigma^0_3 S' \subset \omega^\omega \forall \Pi^0_1 Q \subset 2^\omega : Q <_{\mathrm{w}} \emptyset$
  $\exists \Pi^0_1 R \subset 2^\omega,\ R \equiv_{\mathrm{w}} S' \cup Q$.
- The Embedding Lemma has many consequences,
  and it seems to be useful to prove or disprove the density of $\mathcal{P}_{\mathrm{w}}$.

Some Consequences of The Embedding Lem.

- Thm. $\deg_{\mathrm{w}}(\mathrm{DNR})$, $\deg_{\mathrm{w}}(\mathrm{AED} \cup \mathrm{PA})$, $\deg_{\mathrm{w}}(\mathrm{MLR}^{\emptyset'} \cup \mathrm{PA})$ are in $\mathcal{P}_{\mathrm{w}} \setminus \{\deg_{\mathrm{w}}(2^{\omega}), \deg_{\mathrm{w}}(\mathrm{PA})\}$, $\mathrm{DNR} = \{f \in \omega^{\omega} : \forall e, f(e) \neq \{e\}(e)\}$, AED is the set of all almost everywhere dominating reals, $\mathrm{MLR}^{\emptyset'}$ is the set of all reals Martin-Löf random relative to $\emptyset'$, i.e, the set of 2-random reals.
- Thm(Simpson). The funtion $\phi : \mathcal{R}_{\mathrm{T}} \to \mathcal{P}_{\mathrm{w}}$ s.t. $\phi(\deg_{\mathrm{T}}(A)) = \deg_{\mathrm{w}}(\{A\} \cup \mathrm{PA})$ is an embedding preserving the sup, (therefore $\leq$,) the top and the bot.

Π^0_1 *and* Σ^1_1

- Main Lem.
  $\forall a \in \mathcal{S}_{\mathrm{w}}, b \in \mathcal{P}_{\mathrm{w}}$
  $a < b \Longrightarrow [a, b) \cap \mathcal{P}_{\mathrm{w}} \neq \emptyset$.

  In other words,
  $\forall \Sigma^1_1 S \subset \omega^\omega \forall \Pi^0_1 Q \subset 2^\omega : S <_{\mathrm{w}} Q <_{\mathrm{w}} \emptyset$
  $\exists \Pi^0_1\ R \subset 2^\omega, S \leq_{\mathrm{w}} R <_{\mathrm{w}} Q$.

Cole/Simpson's Lemma

- Lem(Cole/Simpson). $\forall \Pi^0_1\ P, Q \subset \omega^\omega : \{0^\omega\} \not\geq_{\mathrm{w}} Q$
  $\exists \Pi^0_1\ H(P, Q) \simeq P\ \forall g \in H(P, Q),\ \{g\} \not\geq_{\mathrm{w}} Q$.
- Lem′(Cole/Simpson).
  $\forall f \in 2^\omega \forall \Pi^{0,f}_1 P, Q \subset \omega^\omega : \{f\} \not\geq_{\mathrm{w}} Q$
  $\exists \Pi^{0,f}_1 H(f, P, Q) \simeq P\ \forall g \in H(f, P, Q),\ \{f \oplus g\} \not\geq_{\mathrm{w}} Q$.

  Here, note that $P \neq \emptyset \iff H(f, P, Q) \neq \emptyset$.

A Key Lemma

- Lem′(Cole/Simpson).
  $\forall f \in 2^\omega \forall \Pi^{0,f}_1 P, Q \subset \omega^\omega : \{f\} \not\geq_w Q$
  $\exists \Pi^{0,f}_1 H(f, P, Q) \simeq P\ \forall g \in H(f, P, Q),\ \{f \oplus g\} \not\geq_w Q$.
- Lem. $\forall \Sigma^1_1 S, \Pi^0_1 Q \subset \omega^\omega : S <_w Q$
  $\exists \Pi^0_1\ R \subset \omega^\omega, S \leq_w R \not\geq_w Q$.
  $\because$ Define $\Sigma^1_1\ S' = S \cap \{f : \{f\} \not\geq_w Q\}$. (Note $S' \neq \emptyset$.)
  Choose $\Pi^0_1\ P$ s.t. $\forall f\ [f \in S' \iff \exists g,\ f \oplus g \in P]$.
  Let $\Pi^{0,f}_1 P^f = \{g : f \oplus g \in P\}$. (Note $f \in S' \Rightarrow P^f \neq \emptyset$.)
  Define $\Pi^0_1\ R = \{f \oplus g : f \in S', g \in H(f, P^f, Q)\}$.
  Then $R \neq \emptyset$ and $\forall f \oplus g \in R,\ S \leq_w \{f \oplus g\} \not\geq_w Q$.
  Thus $S \leq_w R \not\geq_w Q$. □

Main Lemma

- Lem. $\forall \Sigma^1_1 S, \Pi^0_1 Q \subset \omega^\omega : S <_{\mathrm{w}} Q$
  $\exists \Pi^0_1\ R \subset \omega^\omega, S \leq_{\mathrm{w}} R \not\geq_{\mathrm{w}} Q$.
- The Embedding Lem(Simpson).
  $\forall \Sigma^0_3 S' \subset \omega^\omega \forall \Pi^0_1 Q \subset 2^\omega : Q <_{\mathrm{w}} \emptyset$
  $\exists \Pi^0_1 R \subset 2^\omega,\ R \equiv_{\mathrm{w}} S' \cup Q$.
- Main Lem. $\forall \Sigma^1_1 S \subset \omega^\omega \forall \Pi^0_1 Q \subset 2^\omega : S <_{\mathrm{w}} Q <_{\mathrm{w}} \emptyset$
  $\exists \Pi^0_1\ R \subset 2^\omega, S \leq_{\mathrm{w}} R <_{\mathrm{w}} Q$.
  $\because$ Lem gives $\Pi^0_1\ S' \subset \omega^\omega$ s.t. $S \leq_{\mathrm{w}} S' \not\geq_{\mathrm{w}} Q$.
  Thus $S \leq_{\mathrm{w}} S' \cup Q <_{\mathrm{w}} Q$.
  The Embedding Lem gives the desired R. □

If $\mathcal{P}_{\mathrm{w}}$ is Not Dense

- Main Lem. $\forall \Sigma^1_1 S \subset \omega^\omega \forall \Pi^0_1 Q \subset 2^\omega : S <_{\mathrm{w}} Q <_{\mathrm{w}} \emptyset$ $\exists \Pi^0_1\ R \subset 2^\omega, S \leq_{\mathrm{w}} R <_{\mathrm{w}} Q$.
- Thm. TFAE:
  1. $\exists \Pi^0_1 P, Q \subset 2^\omega : P <_{\mathrm{w}} Q <_{\mathrm{w}} \emptyset$ $\neg \exists \Pi^0_1 R \subset 2^\omega,\ P <_{\mathrm{w}} R <_{\mathrm{w}} Q$.
  2. $\exists \Sigma^1_1 S \subset \omega^\omega \exists \Pi^0_1 Q \subset 2^\omega : S <_{\mathrm{w}} Q <_{\mathrm{w}} \emptyset$ $\neg \exists \Pi^0_1 R \subset 2^\omega,\ S <_{\mathrm{w}} R <_{\mathrm{w}} Q$. □

If $\mathcal{P}_w$ is Dense

- Main Lem. $\forall \Sigma^1_1 S \subset \omega^\omega \forall \Pi^0_1 Q \subset 2^\omega : S <_w Q <_w \emptyset$ $\exists \Pi^0_1\ R \subset 2^\omega, S \leq_w R <_w Q$.
- Thm. $\forall \Pi^0_1 P, Q \subset 2^\omega : P <_w Q <_w \emptyset$, TFAE:
  1. $\exists \Pi^0_1 R \subset 2^\omega,\ P <_w R <_w Q$.
  2. $\exists \Sigma^1_1 S \subset \omega^\omega,\ P <_w S <_w Q$. □

Hyperarithmetic Witness

Let $P, Q \subset 2^\omega$ be nonempty Π^0_1 sets with $P <_{\mathrm{w}} Q$.

- Thm. TFAE:
  1. $\exists \Pi^0_1 R \subset 2^\omega,\ P <_{\mathrm{w}} R <_{\mathrm{w}} Q$.
  2. $\exists \Sigma^1_1 S \subset \omega^\omega,\ P <_{\mathrm{w}} S <_{\mathrm{w}} Q$.
- Cor. If $\exists \Delta^1_1 f \in P, \{f\} \not\geq_{\mathrm{w}} Q$, then 2 holds.
  $\because$ We can prove
  $\forall f : \{f\} \not\geq_{\mathrm{w}} Q \exists \Delta^{0,f}_2 g, \{f\} <_{\mathrm{w}} \{g\} \not\geq_{\mathrm{w}} Q$.
  Thus if f is Δ^1_1, so is g. It is known that $\{g\}$ is Σ^1_1. $\square$
- Cor. If $\#(P \cap \{f : \{f\} \not\geq_{\mathrm{w}} Q\}) \leq \aleph_0$, then 1 holds.
  $\because$ Every countable Σ^1_1 set has only Δ^1_1 elements. $\square$

No Hyperarithmetic Witness

- Main Lem. $\forall \Sigma^1_1 S \subset \omega^\omega \forall \Pi^0_1 Q \subset 2^\omega : S <_w Q <_w \emptyset$
  $\exists \Pi^0_1\ R \subset 2^\omega, S \leq_w R <_w Q$.
- Thm(Jockusch/Soare). $\mu(\{g \in 2^\omega : \{g\} \not\geq_w \mathrm{PA}\}) = 1$
- Thm′(Jockusch/Soare).
  If $\{f\} \not\geq_w \mathrm{PA}$, then $\mu(\{g \in 2^\omega : \{f \oplus g\} \not\geq_w \mathrm{PA}\}) = 1$
- Thm. $\forall \Pi^0_1 P \subset 2^\omega : P <_w \mathrm{PA}\ \exists \Pi^0_1 R \subset 2^\omega$,
  $P \leq_w R <_w \mathrm{PA}$ and $\neg \exists \Delta^1_1 h \in R \cap \{f : \{f\} \not\geq_w \mathrm{PA}\}$.
  $\because$ "g is $\Delta^{1,f}_1$" is Π^1_1 in f and g.
  Define $\Sigma^1_1\ S = \{f \oplus g : f \in P, g \text{ is not } \Delta^{1,f}_1\}$.
  Choose $f \in P$ s.t.$\{f\} \not\geq_w \mathrm{PA}$. Note $\mu(2^\omega \cap \Delta^{1,f}_1) = 0$.
  $\exists g$:non $\Delta^{1,f}_1$ with $\{f \oplus g\} \not\geq_w \mathrm{PA}$.
  Thus $P \leq_w S \cup \mathrm{PA} <_w \mathrm{PA}$.
  Main Lem gives the desired R. □

Thank you!