Theories of concatenation, arithmetic, and undecidability

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Computability Theory and Foundations of Mathematics

Contents

- An introduction for Theories of Concatenation
- Weak theories of concatenation and arithmetic
- Minimal essential undecidability

Back ground and known results

$$\begin{array}{ccc} C^2 & \bowtie & \mathrm{PA} \\ \triangledown & & \triangledown \\ \mathrm{TC} & \bowtie & \mathrm{Q} \end{array}$$

TC : Theory of Concatenation

In A. Grzegorczyk's paper "Undecidability without arithmetization"(2005), he defined a $(\frown, \varepsilon, \alpha, \beta)$-theory TC of concatenation, whose axioms are:

(TC1) $\forall x(x^\frown\varepsilon = \varepsilon^\frown x = x)$ **Axiom for identity**

(TC2) $\forall x\forall y\forall z(x^\frown(y^\frown z) = (x^\frown y)^\frown z)$ **Associativity**

(TC3) Editors Axiom:

$$\forall x\forall y\forall u\forall v(x^\frown y = u^\frown v \rightarrow$$
$$\exists w((x^\frown w = u \wedge y = w^\frown v) \vee (x = u^\frown w \wedge w^\frown y = v)))$$

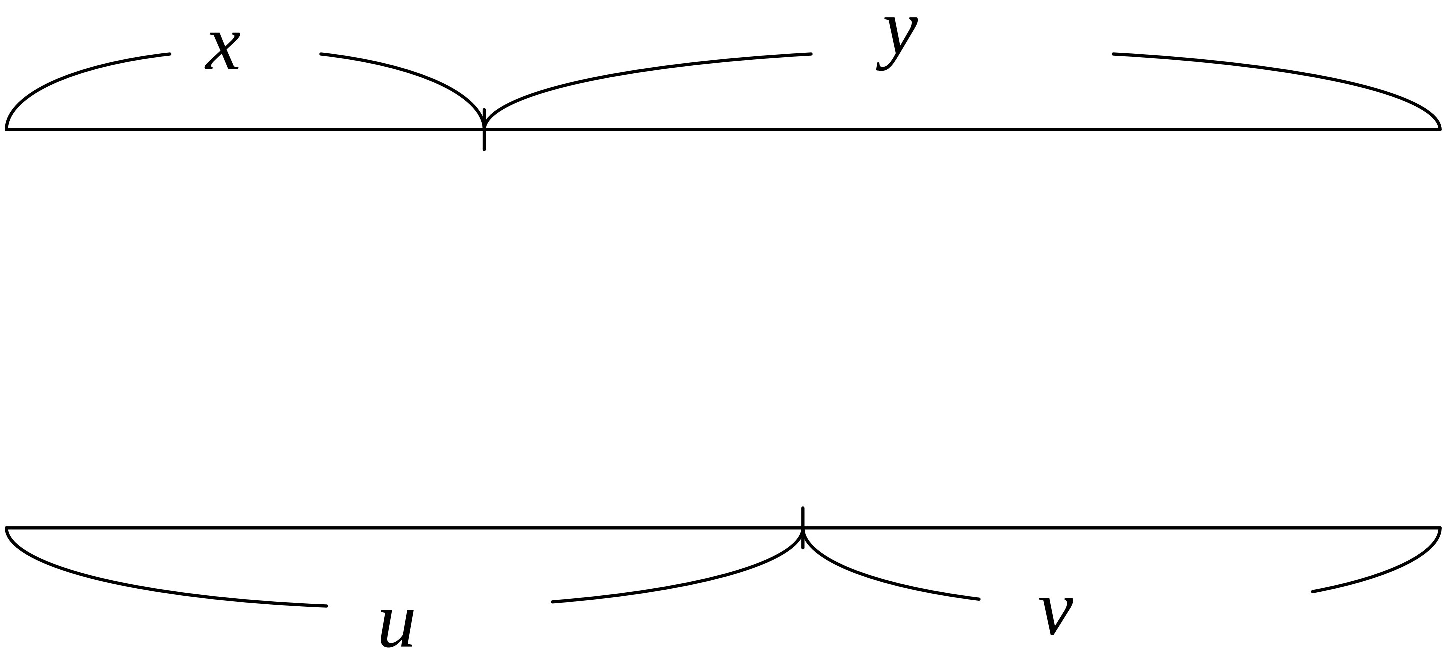
(TC4) $\alpha \neq \varepsilon \wedge \forall x\forall y(x^\frown y = \alpha \rightarrow x = \varepsilon \vee y = \varepsilon)$

(TC5) $\beta \neq \varepsilon \wedge \forall x\forall y(x^\frown y = \beta \rightarrow x = \varepsilon \vee y = \varepsilon)$

(TC6) $\alpha \neq \beta$

About (TC3); editors axiom

If $x^\frown y = u^\frown v$**,**



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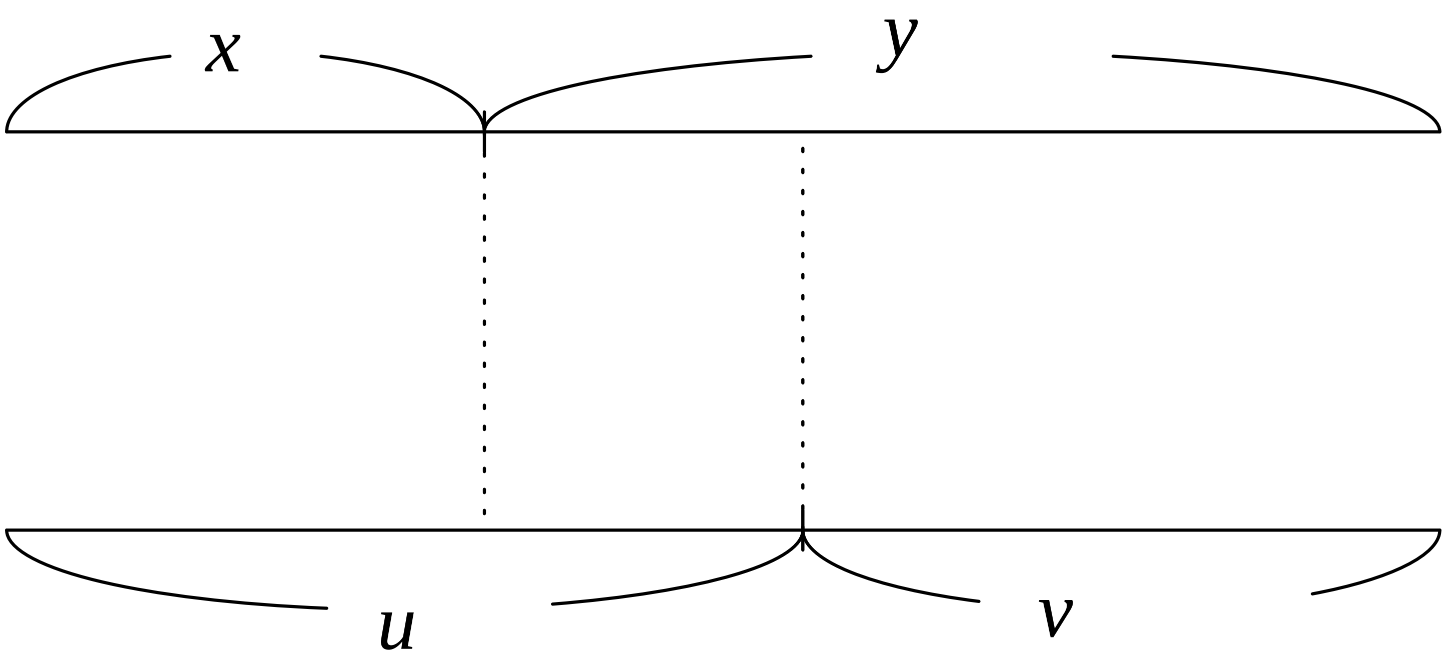

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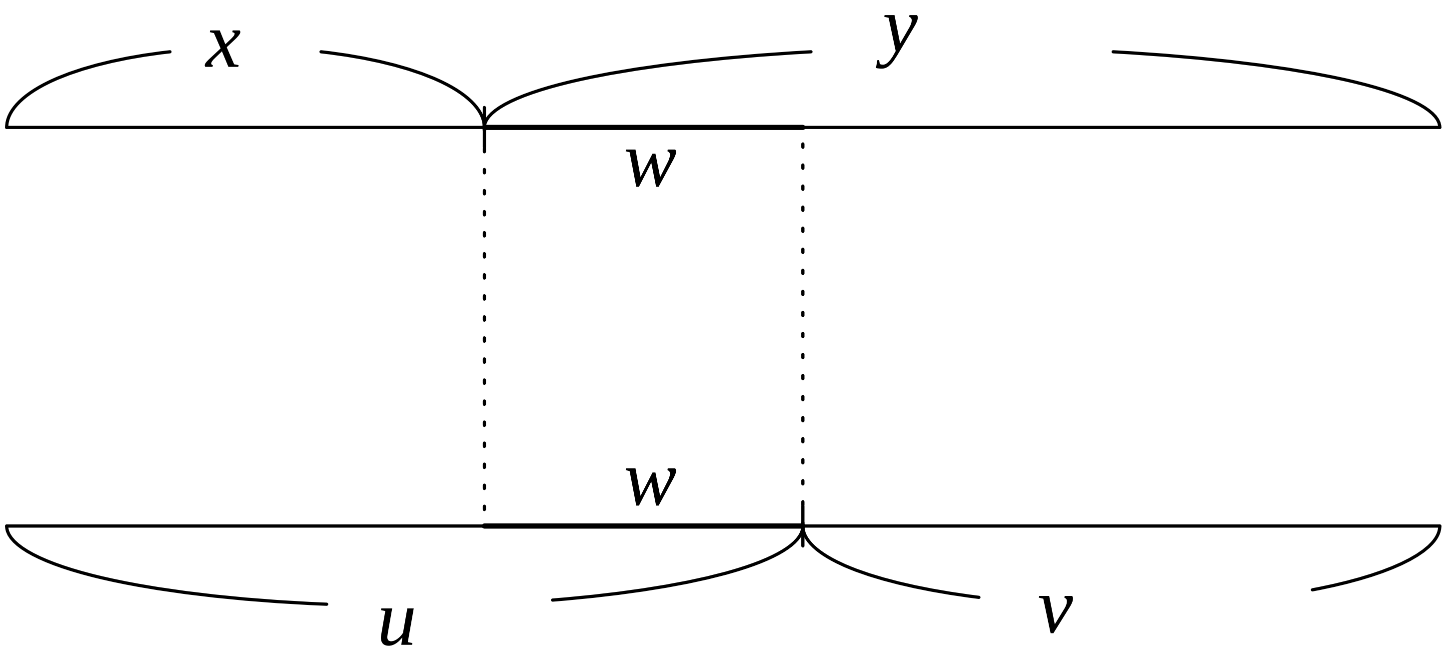

TC : Theory of Concatenation

Definition

- $x \sqsubseteq y \equiv \exists k \exists l ((k^\frown x)^\frown l = y)$
- $x \sqsubseteq_{\mathsf{ini}} y \equiv \exists l \, (x^\frown l = y)$
- $x \sqsubseteq_{\mathsf{end}} y \equiv \exists k \, (k^\frown x = y)$

What can TC prove?

Proposition

TC **proves the following assertions:**

(1) $\forall x(x\alpha \neq \varepsilon \wedge \alpha x \neq \varepsilon)$

(2) $\forall x \forall y(xy = \varepsilon \rightarrow x = \varepsilon \wedge y = \varepsilon)$

(3) $\forall x \forall y(x\alpha = y\alpha \vee \alpha x = \alpha y \rightarrow x = y)$ ***Weak*** **cancellation**

Proposition

TC **cannot prove the following assertions:**

- $\forall x \forall y \forall z(xz = yz \rightarrow x = y)$ **cancellation**

TC and undecidability

Theorem [Grzegorczyk, 2005]

TC **is undecidable.**

Moreover,

Theorem [Grzegorczyk and Zdanowski, 2007]

TC **is essentially undecidable.**

Grzegorczyk and Zdanowski conjectured that
(i) TC **and** Q **are mutually interpretable;**
(ii) TC **is minimal essentially undecidable theory.**

Definition of interpretation

L_1, L_2 **: languages of first order logic.**
A relative translation $\tau : L_1 \to L_2$ **is a pair** $\langle \delta, F \rangle$ **such that**

- δ **is an** L_2**-formula with one free variable.**
- F **maps each relation-symbol** R **of** L_1 **to an** L_2**-formula** $F(R)$**.**

We translate L_1**-formulas to** L_2**-formulas as follows:**

- $(R(x_1, \cdots, x_n))^\tau := F(R)(x_1, \cdots, x_n)$**;**
- $(\cdot)^\tau$ **commutes with the propositional connectives;**
- $(\forall x \varphi(x))^\tau := \forall x(\delta(x) \to \varphi^\tau)$**;**
- $(\exists x \varphi(x))^\tau := \exists x(\delta(x) \wedge \varphi^\tau)$**.**

Definition of interpretation

Definition (relative interpretation)

L_1-theory T is (relatively) interpretable in L_2-theory S, denoted by $S \rhd T$, iff there exists a relative translation $\tau : L_1 \to L_2$ such that
(i) $S \vdash \exists x\, \delta(x)$ and
(ii) for each axiom σ of T, $S \vdash \sigma^\tau$.

Proposition

Let S be a consistent theory.
If $S \rhd T$ and T is essentially undecidable, then S is also essentially undecidable.

The interpretability conserves the essential undecidability.

TC and Q

In 2009, the following results were proved by three ways independently:
Visser and Sterken, Švejdar, and Ganea.

Theorem [2009]

TC **interprets** Q**. (Hence** $\mathsf{TC} \rhd\lhd \mathsf{Q}$**.)**

Here, Q **is Robinson's arithmetic, whose language is** $(+, \cdot, 0, S)$

(Q1) $\forall x \forall y (S(x) = S(y) \to x = y)$

(Q2) $\forall x (S(x) \neq 0)$

(Q3) $\forall x (x + 0 = x)$

(Q4) $\forall x \forall y (x + S(y) = S(x + y))$

(Q5) $\forall x (x \cdot 0 = 0)$

(Q6) $\forall x \forall y (x \cdot S(y) = x \cdot y + x)$

(Q7) $\forall x (x \neq 0 \to \exists y (x = S(y)))$

Q **is essentially undecidable and finitely axiomatizable.**

Theory C^2 and Peano arithmetic PA

The theory C^2 of concatenation consists of TC plus the following induction:

$$\varphi(\varepsilon) \wedge \forall x\,(\varphi(x) \to \varphi(x^\frown\alpha) \wedge \varphi(x^\frown\beta)) \to \forall x\,\varphi(x).$$

Here, φ is a $(^\frown, \varepsilon, \alpha, \beta)$-formula.

Then, Ganea proved that

Theorem [Ganea, 2009]

C^2 **and** PA **are mutually interpretable.**

Part I

A weak theory WTC of concatenation and mutual interpretability with R

Arithmetic R (Mostowski-Robinson-Tarski, 1953)

$(+,\cdot,0,1,\leq)$-theory R

For each $n,m\in\omega$**, (** $\overline{n}$ **represents** $\underbrace{1+\cdots+1}_{n}$ **)**

(R1) $\overline{n}+\overline{m}=\overline{n+m}$

(R2) $\overline{n}\cdot\overline{m}=\overline{n\cdot m}$

(R3) $\overline{n}\neq\overline{m}$ $($**if** $n\neq m)$

(R4) $\forall x\left(x\leq\overline{n}\rightarrow x=\overline{0}\vee x=\overline{1}\vee\cdots\vee x=\overline{n}\right)$

(R5) $\forall x(x\leq\overline{n}\vee\overline{n}\leq x)$

* R **is Σ_1-complete and essentially undecidable.**
* $\mathsf{R}\not\rhd\mathsf{Q}$**, since** Q **is finitely axiomatizable.**

Arithmetic R_0 (Cobham, 1960's)

$(+,\cdot,0,1,\leq)$-theory R_0

For each $n,m \in \omega$,

(R1) $\overline{n}+\overline{m}=\overline{n+m}$

(R2) $\overline{n}\cdot\overline{m}=\overline{n\cdot m}$

(R3) $\overline{n}\neq\overline{m}$ $($**if** $n\neq m)$

(R4') $\forall x\left(x\leq\overline{n}\leftrightarrow x=\overline{0}\vee x=\overline{1}\vee\cdots\vee x=\overline{n}\right)$

*** R_0 interprets R by translating '$\leq$' by '$\lessdot$' as follows:**

$$x\lessdot y\equiv[0\leq y\wedge\forall u\,(u\leq y\wedge u\neq y\rightarrow u+1\leq y)]\rightarrow x\leq y.$$

*** R_0 is *minimal* theory which is Σ_1-complete and essentially undecidable.**

Arithmetic R_1 (Jones and Shepherdson, 1983)

$(+,\cdot,0,1,\leq)$**-theory** R_1

For each $n, m \in \omega$**,**

(R2) $\overline{n}\cdot\overline{m} = \overline{n\cdot m}$

(R3) $\overline{n} \neq \overline{m}$ $\;($**if** $\;n \neq m)$

(R4') $\forall x\left(x \leq \overline{n} \leftrightarrow x = \overline{0} \vee x = \overline{1} \vee \cdots \vee x = \overline{n}\right)$

***R_1 interprets R_0 by J. Robinson's definition of addition in terms of multiplication.**

***R_1 is *minimal* theory which is essentially undecidable.**

WTC: Weak Theory of Concatenation

$(\frown, \varepsilon, \alpha, \beta)$**-theory** WTC **has the following axioms: for each** $u \in \{\alpha, \beta\}^*$**,**

(WTC1) $\forall x \underline{\sqsubseteq} \underline{u} (x \frown \varepsilon = \varepsilon \frown x = x)$**;**

(WTC2) $\forall x \forall y \forall z [[x \frown (y \frown z) \underline{\sqsubseteq} \underline{u} \vee (x \frown y) \frown z \underline{\sqsubseteq} \underline{u}] \longrightarrow x \frown (y \frown z) = (x \frown y) \frown z]$**;**

(WTC3) $\forall x \forall y \forall s \forall t [(x \frown y = s \frown t \wedge x \frown y \underline{\sqsubseteq} \underline{u}) \longrightarrow \exists w ((x \frown w = s \wedge y = w \frown t) \vee (x = s \frown w \wedge w \frown y = t))]$**;**

(WTC4) $\alpha \neq \varepsilon \wedge \forall x \forall y (x \frown y = \alpha \rightarrow x = \varepsilon \vee y = \varepsilon)$**;**

(WTC5) $\beta \neq \varepsilon \wedge \forall x \forall y (x \frown y = \beta \rightarrow x = \varepsilon \vee y = \varepsilon)$**;**

(WTC6) $\alpha \neq \beta$**.**

WTC: Weak Theory of Concatenation

Here, $\{\alpha,\beta\}^*$ is a set of finite strings over $\{\alpha,\beta\}$, including empty string ε. Let $\{\alpha,\beta\}^+ := \{\alpha,\beta\}^* \setminus \{\varepsilon\}$.
For each $u \in \{\alpha,\beta\}^*$, we represent u in theories as $\underline{u}$ by adding parentheses from *left*. For example, $\underline{\alpha\alpha\beta\alpha} = ((\alpha\alpha)\beta)\alpha$. We call each u $(\in \{\alpha,\beta\}^*)$ standard string.

Definition

- $x \sqsubseteq y \equiv (x = y) \vee \exists k \exists l\, [kx = y \vee xl = y \vee (kx)l = y \vee k(xl) = y]$
- $x \sqsubseteq_{\mathsf{ini}} y \equiv (x = y) \vee \exists l\, (xl = y)$
- $x \sqsubseteq_{\mathsf{end}} y \equiv (x = y) \vee \exists k\, (kx = y)$

Σ_1-completeness of WTC

Lemma

WTC **proves the following assertion:**

$$\forall x \left(x \sqsubseteq \underline{u} \leftrightarrow \bigvee_{v \sqsubseteq u} x = \underline{v} \right).$$

Theorem

WTC **is Σ_1-complete, that is, for each Σ_1-sentence φ, if $\{\alpha, \beta\}^* \vDash \varphi$ then $\mathsf{WTC} \vdash \varphi$.**

$\{\alpha, \beta\}^*$ is a standard model of TC.

WTC **interprets** R

From now on, we consider the translation of R into WTC.

translation of $0, 1, +$

We translate $0, 1, +$ **as follows:**

- $0 \Rightarrow \varepsilon$;
- $1 \Rightarrow \alpha$;
- $x + y \Rightarrow x^\frown y$;
- $x \leq y \Rightarrow \exists z\,(x^\frown z = y)$.

To translate the product, we have to make it total on ω. To do this, we consider notion, "witness for product".

WTC **interprets** R

An idea for the definition of witness

Witness w **for** 2×3 **is as follows:**

$$w = \beta\beta\beta\beta\beta\alpha\beta\alpha\alpha\beta\beta\alpha\alpha\beta(\alpha\alpha)(\alpha\alpha)\beta\beta\alpha\alpha\alpha\beta(\alpha\alpha)(\alpha\alpha)(\alpha\alpha)\beta\beta$$

This is from the following interpretation of 2×3**:**

$$(0,0) \longrightarrow (1,2) \longrightarrow (2,2+2) \longrightarrow (3,2+2+2).$$

That is, 2×3 **is interpreted as adding** 2 **three times.**

By the help of above idea, we can represent the relation “ w ***is a witness for product of*** x ***and*** y **” by a formula** $\mathrm{PWitn}(x,y,w)$**.**

WTC **interprets** R

Translation of product

We translate the multiplication "$x \times y = z$**" by**
$(\exists ! w\, \mathrm{PWitn}(x,y,w) \wedge \beta\beta y \beta z \beta\beta \sqsubseteq_{\mathsf{end}} w) \vee$
$(\neg(\exists ! w\, \mathrm{PWitn}(x,y,w))) \wedge z = 0$**.**

Lemma (uniqueness of the witness on ω)

For each $u, v \in \{\alpha\}^*$**, there exists** $w \in \{\alpha,\beta\}^*$ **such that** WTC **proves**
$\mathrm{PWitn}(\underline{u},\underline{v},\underline{w}) \wedge \forall w' (\mathrm{PWitn}(\underline{u},\underline{v},w') \rightarrow \underline{w} = w')$.

Theorem

WTC **interprets** R**.**

R interprets WTC

Conversely, we can prove that R interprets WTC, by applying the Visser's following theorem:

Visser's theorem (2009)

T is interpretable in R iff T is locally finitely satisfiable

Here, a theory T is locally finitely satisfiable iff any finite subtheory of T has a finite model.
Since WTC is locally finitely satisfiable, we can get the following result:

Corollary

R interprets WTC.

Conclusion of part I

Theorem

WTC **and** R **are mutually interpretable.**

Corollary

(1) WTC **is essentially undecidable.**

(2) WTC **interprets** T **iff** T **is locally finitely satisfiable.**

(3) WTC **cannot interpret** TC**.**

(4) WTC_2 **and** WTC_n $(n \geq 2)$ **are mutually interpretable.**

Here, WTC_n **is** WTC **with** n**-th single-letters. (4) is from** $\mathsf{WTC}_2 \rhd \mathsf{R} \rhd \mathsf{WTC}_n \rhd \mathsf{WTC}_2$**.**

Part II

Minimal essential undecidability and variations of WTC

Minimal essential undecidability

Question

Is WTC *minimal* essentially undecidable ?

Here, minimal essentially undecidable means if one omits one axiom from WTC, then the resulting theory is no longer essentially undecidable. Again, WTC is: for each $u \in \{\alpha,\beta\}^*$

(WTC1) $\forall x \sqsubseteq \underline{u}\,(x^\frown\varepsilon = \varepsilon^\frown x = x)$**;**

(WTC2) $\forall x\forall y\forall z[[x^\frown(y^\frown z) \sqsubseteq \underline{u} \vee (x^\frown y)^\frown z \sqsubseteq \underline{u}] \rightarrow x^\frown(y^\frown z) = (x^\frown y)^\frown z]$**;**

(WTC3) $\forall x\forall y\forall s\forall t[(x^\frown y = s^\frown t \wedge x^\frown y \sqsubseteq \underline{u}) \rightarrow \exists w((x^\frown w = s \wedge y = w^\frown t) \vee (x = s^\frown w \wedge w^\frown y = t))]$**;**

(WTC4) $\alpha \neq \varepsilon \wedge \forall x\forall y(x^\frown y = \alpha \rightarrow x = \varepsilon \vee y = \varepsilon)$**;**

(WTC5) $\beta \neq \varepsilon \wedge \forall x\forall y(x^\frown y = \beta \rightarrow x = \varepsilon \vee y = \varepsilon)$**;**

(WTC6) $\alpha \neq \beta$**.**

Minimal essential undecidability

Proposition

$\mathsf{WTC}-$**(WTC** k**) (**$k = 3, 4, 5, 6$**) is not essentially undecidable.**

We can find a decidable consistent extension of each $\mathsf{WTC}-$**(WTC** k**) (**$k = 3, 4, 5, 6$**). Hence remaining question is**

$\mathsf{WTC}-$**(WTC** k**) (**$k = 1, 2$**) is essentially undecidable ?**

Minimal essential undecidability

Proposition

$\mathsf{WTC}-$**(WTC** k**)** **(**$k = 3, 4, 5, 6$**) is not essentially undecidable.**

We can find a decidable consistent extension of each $\mathsf{WTC}-$**(WTC** k**)** **(**$k = 3, 4, 5, 6$**). Hence remaining question is**

$\boxed{\mathsf{WTC}-\textbf{(WTC}\, k\textbf{) (}k = 1, 2\textbf{) is essentially undecidable ?}}$

We have proved the following:

Theorem (with O. Yoshida)

$\mathsf{WTC}-$**(WTC**1**) can interpret** WTC**.**
Hence, $\mathsf{WTC}-$**(WTC**1**) is still essentially undecidable.**

WTC$-$**(WTC1)** $\rhd \lhd$ WTC

This is proved by the following two lemmas.

Lemma

For each $u \in \{\alpha, \beta\}^*$**,** WTC **- (WTC1) proves** $\underline{u}\varepsilon = \varepsilon\underline{u} = \underline{u}$**.**

$\Rightarrow$ **Without (WTC1), axiom for identity, we can prove that the empty string works well, as an identity element, for at least all standard strings.**

Lemma

WTC **- (WTC1)** $\vdash \forall x\,(x \sqsubseteq \underline{u} \wedge \exists x'\,(x = (\varepsilon x')\varepsilon) \rightarrow \bigvee_{v \sqsubseteq u} x = \underline{v})$**.**

Although we do not know whether WTC$-$**(WTC1) can prove** $\forall x\,(x \sqsubseteq \underline{u} \rightarrow \bigvee_{v \sqsubseteq u} x = \underline{v})$ **or not, the above Lemma is strong enough to interpret** WTC **into** WTC$-$**(WTC1).**

$\mathsf{WTC}-$(**WTC**1) $\rhd\lhd$ WTC

Then, we interpret WTC **in** WTC **- (WTC1) as follows:**

$\boxed{\textbf{Domain}}$ $\delta(x) \equiv x = \alpha \vee \exists x' \, (x = (\beta x')\varepsilon)$.

Remark that if $(\beta x')\varepsilon$ ***is standard, then*** $(\beta x')\varepsilon = \beta((\varepsilon x')\varepsilon)$.

$\boxed{\textbf{Constants}}$ $\varepsilon \Rightarrow \beta$, $\alpha \Rightarrow \beta\alpha$, $\beta \Rightarrow \beta\beta$.

$\boxed{x^\frown y = z}$ **Let** $\Omega(x,y) \equiv \exists! x' \exists! y' \, (x = (\beta x')\varepsilon \wedge y = (\beta y')\varepsilon)$.

Then we translate concatenation as $\mathrm{Conc}(x,y,z) \equiv$

$$x = \alpha \vee y = \alpha \to z = \alpha$$
$$\wedge \Omega(x,y) \to \exists x' \exists y' \, [x = (\beta x')\varepsilon \wedge y = (\beta y')\varepsilon \wedge z = (\beta((x'\varepsilon)y'))\varepsilon]$$
$$\wedge \text{o.w.} \to z = \alpha.$$

Lemma

For each $w \in \{\alpha,\beta\}^*$, WTC **- (WTC1) can prove that if** $\mathrm{Conc}(x,y,\beta \underline{w})$**, then** x **and** y **are also standard.**

$\mathsf{WTC}-$**(WTC**1**)** $\rhd \lhd$ WTC

Question

Is $\mathsf{WTC}-$**(WTC1) minimal essentially undecidable ?**

$\mathsf{WTC}-$**(WTC1)** $\rhd \lhd$ WTC

Question

Is $\mathsf{WTC}-$(WTC1) minimal essentially undecidable ?

Theorem (K. Higuchi)

$\mathsf{WTC}-$(WTC1) is interpretable in $\mathsf{S2S}$.

Here, $\mathsf{S2S}$ is a *monadic second-order logic* whose language is $L = \{\mathrm{S}_0, \mathrm{S}_1, (\mathrm{P_a})_{a \in A}\}$. $\mathrm{S}_0, \mathrm{S}_1$ are two successors and P_a's are unary predicates. Then, $\mathsf{S2S} := \{\varphi \mid \varphi \text{ is an } L\text{-sentence } \& \ \{0,1\}^* \models \varphi\}$. $\mathsf{S2S}$ is proved to be decidable by M. O. Rabin (1969).

Theorem

$\mathsf{WTC}-$(WTC1) is minimal essentially undecidable theory.

$\mathsf{TC}^{-\varepsilon}$

On the other hand, we can consider the theory of concatenation without empty string: $(\frown, \alpha, \beta)$**-theory** $\mathsf{TC}^{-\varepsilon}$ **has the following axioms:**

(TC$^{-\varepsilon}$1) $\forall x \forall y \forall z (x^\frown(y^\frown z) = (x^\frown y)^\frown z)$ **Associativity**

(TC$^{-\varepsilon}$2) Editors Axiom:

$$\forall x \forall y \forall s \forall t\, (x^\frown y = s^\frown t \rightarrow (x = s \wedge y = t) \vee$$
$$\exists w\, ((x^\frown w = s \wedge y = w^\frown t) \vee (x = s^\frown w \wedge w^\frown y = t)))$$

(TC$^{-\varepsilon}$3) $\forall x \forall y\, (\alpha \neq x^\frown y)$

(TC$^{-\varepsilon}$4) $\forall x \forall y\, (\beta \neq x^\frown y)$

(TC$^{-\varepsilon}$5) $\alpha \neq \beta$

$\mathsf{WTC}^{-\varepsilon}$

A weak version $\mathsf{WTC}^{-\varepsilon}$ **of** $\mathsf{TC}^{-\varepsilon}$ **has the following axioms: for each** $u \in \{\alpha, \beta\}^+$**,**

(WTC$^{-\varepsilon}$1) $\forall x \forall y \forall z\, [[x^\frown(y^\frown z) \sqsubseteq u \vee (x^\frown y)^\frown z \sqsubseteq u]$
$\rightarrow x^\frown(y^\frown z) = (x^\frown y)^\frown z]$**;**

(WTC$^{-\varepsilon}$2) $\forall x \forall y \forall s \forall t\, [(x^\frown y = s^\frown t \wedge x^\frown y \sqsubseteq u) \rightarrow$
$(x = y) \wedge (s = t) \vee$
$\exists w\, ((x^\frown w = s \wedge y = w^\frown t) \vee (x = s^\frown w \wedge w^\frown y = t))]$**;**

(WTC$^{-\varepsilon}$3) $\forall x \forall y\, (x^\frown y \neq \alpha)$**;**

(WTC$^{-\varepsilon}$4) $\forall x \forall y\, (x^\frown y \neq \beta)$**;**

(WTC$^{-\varepsilon}$5) $\alpha \neq \beta$**.**

For this theory, we proved the following:

$\mathsf{WTC}^{-\varepsilon} \rhd \lhd \mathsf{WTC}$

Proposition

$\mathsf{WTC}^{-\varepsilon}$ **and** WTC **are mutually interpretable.**
Hence $\mathsf{WTC}^{-\varepsilon}$ **is essentially undecidable.**

$\mathsf{WTC} \rhd \mathsf{WTC}^{-\varepsilon}$ **is easy. We interpret** WTC **in** $\mathsf{WTC}^{-\varepsilon}$ **as:**

$\boxed{\textbf{Domain}}$ $\delta(x) \equiv x = \alpha \vee x = \beta \vee \exists x' \,(x = \beta x')$**.**

$\boxed{\textbf{Constants}}$ $\varepsilon \Rightarrow \beta$**,** $\alpha \Rightarrow \beta\alpha$**,** $\beta \Rightarrow \beta\beta$**.**

$\boxed{x \frown y = z}$ **Let** $\Omega(x,y) \equiv \exists! x' \,\exists! y' \,(x = \beta x' \wedge y = \beta y')$**, and translate the concatenation by** $\mathrm{Conc}(x,y,z) \equiv$
$[x = \alpha \vee y = \alpha \to z = \alpha] \wedge [x = \beta \to z = y] \wedge [y = \beta \to z = x] \wedge$
$[\Omega(x,y) \to \exists x' \,\exists y' \,(x = \beta x' \wedge y = \beta y' \wedge z = \beta(x'y'))] \wedge$
$[\text{o.w.} \ \to z = \alpha]$**.**

$\mathsf{WTC}^{-\varepsilon}$ is minimal essentially undecidable

Theorem

$\mathsf{WTC}^{-\varepsilon}$ **is minimal essentially undecidable.**

This result partially contributes the following question by Grzegorczyk and Zdanowski:

Question

Is $\mathsf{TC}^{-\varepsilon}$ minimal essentially undecidable ?

The remaining part of the question is the essential undecidability of $\mathsf{TC}^{-\varepsilon}$–(TC$^{-\varepsilon}1$), that is, TC without associative law. We can easily find an decidable extension of each $\mathsf{TC}^{-\varepsilon}$–(TC$^{-\varepsilon}k$), ($k = 2, 3, 4, 5$).

Variations of WTC**:** $\mathsf{WTC}+$**(TC1)** $+$ **(TC2)** $\rhd\lhd$ WTC

Recall that

(TC1) $\forall x\,(x^\frown\varepsilon=\varepsilon^\frown x=x)$

(TC2) $\forall x\forall y\forall z\,(x^\frown(y^\frown z)=x^\frown(y^\frown z))$

(TC3) $\forall x\forall y\forall s\forall t\,[(x^\frown y=s^\frown t)\rightarrow$

$\exists w((x^\frown w=s\wedge y=w^\frown t)\vee(x=s^\frown w\wedge w^\frown y=t))]$

Proposition

WTC **interprets** $\mathsf{WTC}+$**(TC1)** $+$ **(TC2)**

Because $\mathsf{WTC}+$**(TC1)** $+$ **(TC2) is locally finitely satisfiable.**

Proposition

WTC **can not interpret** $\mathsf{WTC}+$**(TC3).**

Because $\mathsf{WTC}+$**(TC3) is not locally finitely satisfiable.**

Conclusion of Part II

The following are mutually interpretable ($n \geq 2$):

$\mathsf{WTC}_n + (\text{Identity}) + (\text{Assoc})$

$\mathsf{WTC}_n + (\text{Identity})$

$\mathsf{WTC}_n + (\text{Assoc})$ $\qquad$ $\mathsf{WTC}_n^{-\varepsilon} + (\text{Assoc})$

WTC_n $\qquad$ $\mathsf{WTC}_n^{-\varepsilon}$

$\mathsf{WTC}_n - (\text{WTC1})$

Theorem

$\mathsf{WTC}-$**(WTC1),** $\mathsf{WTC}^{-\varepsilon}$ **is minimal essentially undecidable.**

Questions

(1) **Is** WTC**-(Identity)** Σ_1**-complete ?**
$\Rightarrow$ **Our conjecture is NO.**

(2) $\mathsf{WTC}+$ **(Editors Axiom)** $\rhd \mathsf{TC}$ **?**
$\Rightarrow$ **Our conjecture is YES.**

(3) **Are there some natural theory** T **such that** $\mathsf{TC} \rhd T \rhd \mathsf{WTC}$ **and** $\mathsf{WTC} \not\rhd T$ **and** $T \not\rhd \mathsf{TC}$ **?**

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WTC **interprets** R

Definition of "Good"

We define the formula Good(x) **as follows:**

$$\mathrm{Good}(x) \equiv \mathrm{ID}(x) \wedge \mathrm{AS}(x) \wedge \mathrm{EA}(x), \quad \text{where}$$

- $\mathrm{ID}(x) \equiv \forall s \sqsubseteq x(s^\frown \varepsilon = \varepsilon^\frown s = s)$**;**
- $\mathrm{AS}(x) \equiv \forall s_0 \forall s_1 \forall s_2[[s_0{}^\frown(s_1{}^\frown s_2) \sqsubseteq x \vee (s_0{}^\frown s_1)^\frown s_2 \sqsubseteq x] \rightarrow s_0{}^\frown(s_1{}^\frown s_2) = (s_0{}^\frown s_1)^\frown s_2]$
- $\mathrm{EA}(x) \equiv \forall s_0 \forall s_1 \forall t_0 \forall t_1[(s_0{}^\frown s_1 = t_0{}^\frown t_1 \wedge s_0{}^\frown s_1 \sqsubseteq x) \rightarrow \exists w((s_0{}^\frown w = t_0 \wedge s_1 = w^\frown t_1) \vee (s_0 = t_0{}^\frown w \wedge w^\frown s_1 = t_1))]$

WTC interprets R

Properties of Good

(1) **For each** $u \in \{\alpha, \beta, \gamma\}^*, \mathsf{WTC} \vdash \mathrm{Good}(\underline{u})$**;**

WTC **proves the following assertions:**

(2) $\forall x(\mathrm{Good}(x) \to \forall y \sqsubseteq x\, \mathrm{Good}(y))$**, that is** Good **is closed under taking substrings.**

WTC **interprets** R

To translate the product, we define "witness for product".
First, we define a notion "number strings" as follows:

Definition of "Num"

We define the formula $\mathrm{Num}(x)$ **as follows:**

$$\mathrm{Num}(x) \equiv \forall y((y \sqsubseteq x \wedge y \neq \varepsilon) \to \alpha \sqsubseteq_{\mathsf{end}} y).$$

Fact

For each $u \in \{\alpha\}^*, \mathsf{WTC} \vdash \mathrm{Num}(\underline{u})$**.**

Definition of PWitn

We define a formula $\mathrm{PWitn}(x,y,w)$ **as follows:**

(i) $\mathrm{Num}(x) \wedge \mathrm{Num}(y) \wedge \mathrm{Good}(w)$;

(ii) $\beta\gamma\beta \sqsubseteq_{\mathsf{ini}} w$;

(iii) $\exists \mathrm{z}(\mathrm{Num}(z) \wedge \beta y \gamma \mathrm{z} \beta \sqsubseteq_{\mathsf{end}} w)$;

(iv) $\forall p \forall \mathrm{z}(\mathrm{Num}(z) \wedge p\beta y \gamma \mathrm{z} \beta = w \rightarrow \forall \mathrm{z}'(\mathrm{Num}(z') \rightarrow \neg(\beta y \gamma \mathrm{z}' \beta \sqsubseteq p\beta))$;

(v) $\forall p \forall q \forall s_2 \forall t_2[(\mathrm{Num}(s_2) \wedge \mathrm{Num}(t_2) \wedge p\beta s_2 \gamma t_2 \beta q = w \wedge p \neq \varepsilon)$
$\rightarrow (\exists s_1 \exists t_1(\mathrm{Num}(s_1) \wedge \mathrm{Num}(t_1) \wedge s_2 = s_1\alpha \wedge t_2 = t_1 x \wedge \beta s_1 \gamma t_1 \beta \sqsubseteq_{\mathsf{end}} p\beta))]$;

(vi) $\forall p \forall q \forall s \forall t((\mathrm{Num}(s_1) \wedge \mathrm{Num}(t_1) \wedge p\beta s \gamma t \beta q = w \wedge q \neq \varepsilon) \rightarrow \beta s \alpha \gamma t x \beta \sqsubseteq_{\mathsf{ini}} \beta q)$.

WTC **interprets** R

$$\mathrm{PWitn}(x, y, w)$$

WTC **interprets** R

$$\mathrm{PWitn}(x, y, w)$$

condition (ii)

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$$\mathrm{PWitn}(x, y, w)$$

condition (iii)

$\beta y \gamma z \beta$ **for some** z

w

WTC **interprets** R

$$\mathrm{PWitn}(x, y, w)$$

condition (iv)

WTC **interprets** R

Translation of product

We translate the multiplication "$x \times y = z$" into the formula $\mathrm{M}(x,y,z)$ **as follows:**

$$\mathrm{M}(x,y,z) \equiv (\exists! w \mathrm{PWitn}(x,y,w) \wedge \gamma \mathrm{z} \boldsymbol{\beta} \sqsubseteq_{\mathsf{end}} w) \vee \\ (\neg(\exists! w \mathrm{PWitn}(x,y,w))) \wedge z = 0.$$

WTC **interprets** R

Main theorem

For each $u, v \in \{a\}^+$**, there exists** $w \in \{a, b, c\}^+$ **such that** WTC **proves**

$$\mathrm{PWitn}(\underline{u}, \underline{v}, \underline{w}) \wedge \forall w' (\mathrm{PWitn}(\underline{u}, \underline{v}, w') \rightarrow \underline{w} = w').$$

In what follows, we see the each steps of the proof of this main theorem.

WTC interprets R

Lemma

WTC **proves the following assertions:**

(1) $\mathrm{Good}(x\beta s) \wedge x\beta s = y\beta t \wedge \neg(\beta \sqsubseteq s) \wedge \neg(\beta \sqsubseteq t)$
$\longrightarrow (x = y \wedge s = t)$.

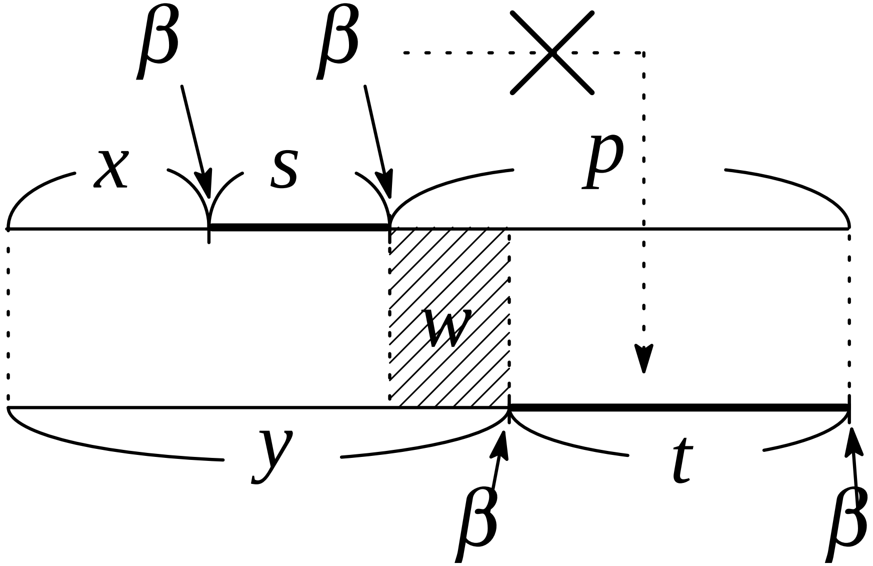
(2) $\mathrm{Good}(x\beta s\beta p) \wedge x\beta s\beta p = y\beta t\beta \wedge \neg(\beta \sqsubseteq s) \wedge \neg(\beta \sqsubseteq t)$

$$\longrightarrow \begin{cases} p \neq \varepsilon \longrightarrow \exists w(x\beta s\beta w = y\beta \wedge wt\beta = p) \vee \\ p = \varepsilon \longrightarrow (x = y \wedge s = t). \end{cases}$$

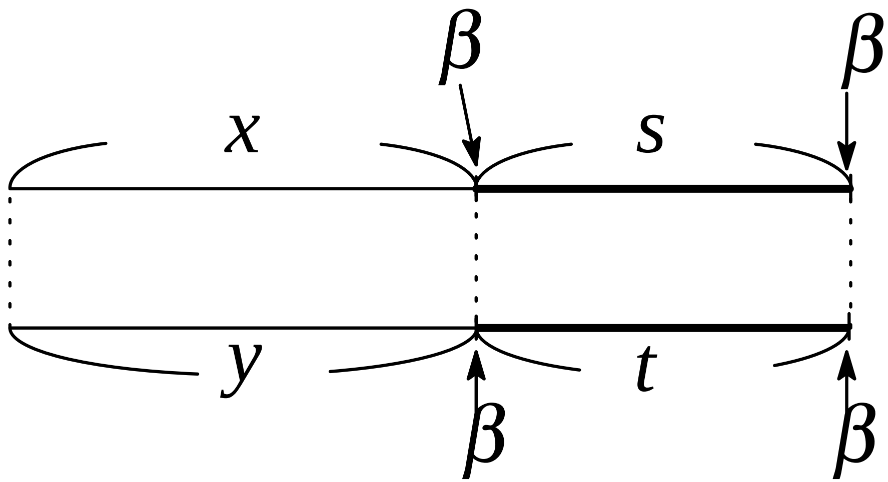
WTC **interprets** R

If $x\beta s\beta p = y\beta t\beta$**,**

(a) $p \neq \varepsilon$



(b) $p = \varepsilon$



WTC **interprets** R

Existence of the witness

Fix $u \in \{a\}^+$.

We can prove the existence of the witness $w \in \{a,b,c\}^+$ **by the meta-induction on the length of** $v \in \{a\}^+$.

WTC **interprets** R

To prove the uniqueness of the witness, we prove thie by the following two steps: Fix $u, v \in \{a\}^+$ **and let** $w \in \{a,b,c\}^+$ **be some witness for** u, v**. In** WTC**, let** w' **be such that** $\mathrm{PWitn}(\underline{u}, \underline{v}, w')$**. Then,**

Step 1

(1) **For each** $k, l \in \{a\}^+$**,**
$\mathsf{WTC} \vdash \forall p(p\beta\underline{k}\gamma\underline{l}\beta \sqsubseteq_{\mathsf{ini}} \underline{w} \to p\beta\underline{k}\gamma\underline{l}\beta \sqsubseteq_{\mathsf{ini}} w')$**;**

(2) $\mathsf{WTC} \vdash \underline{w} \sqsubseteq_{\mathsf{ini}} w'$**.**

WTC **interprets** R

$$\underline{w} \qquad \frac{\beta\gamma\beta}{}$$

$$w'$$

WTC **interprets** R

$\underline{w}$ $\qquad\qquad$ $\beta\gamma\beta$

w' $\qquad\qquad$ $\beta\gamma\beta$

WTC interprets R

$\beta\alpha\gamma\underline{u}\beta$

$\underline{w}$

w'

WTC **interprets** R

$\underline{w}$ $\qquad$ $\beta\alpha\gamma\underline{u}\beta$

$\beta\alpha\gamma\underline{u}\beta$

w'

WTC **interprets** R

$\beta\alpha\alpha\gamma\underline{uu}\beta$

$\underline{w}$

w'

WTC **interprets** R

$\beta\alpha\alpha\gamma\underline{uu}\beta$

$\underline{w}$

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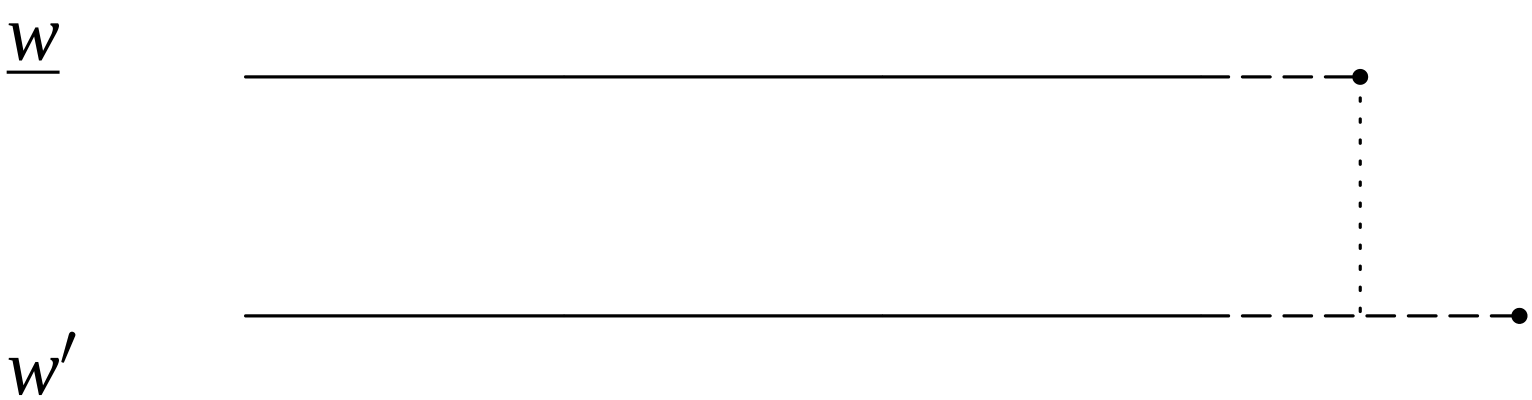
w'

WTC **interprets** R



WTC **interprets** R

Step 2

$\mathsf{WTC} \vdash \underline{w} = w'$.

We prove this by way of contradiction. Let us assume that $\exists q(\underline{w}q = w' \wedge q \neq \varepsilon)$**.**

WTC **interprets** R

Step 2

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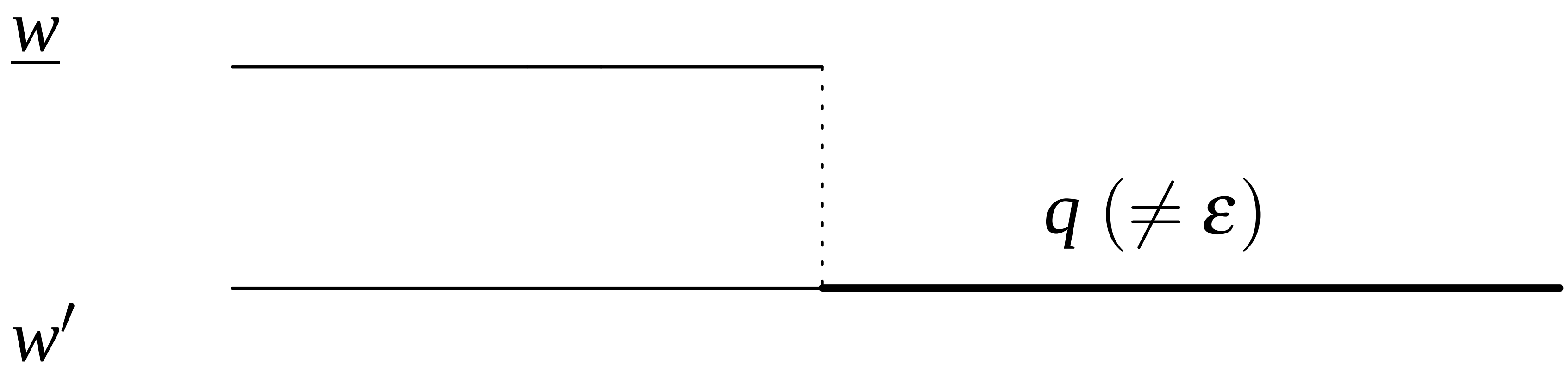

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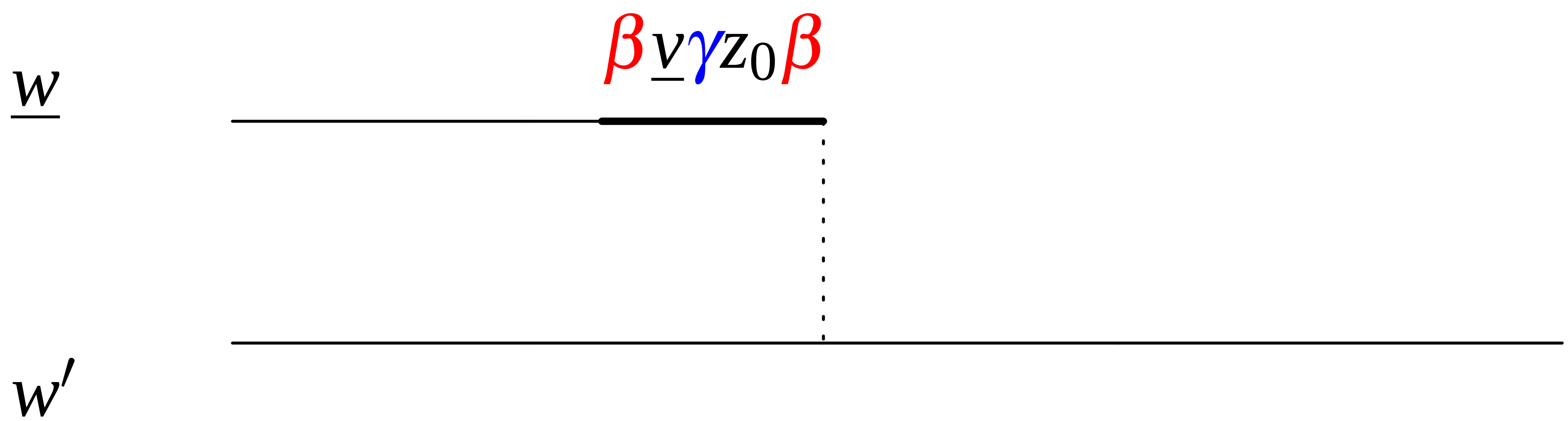

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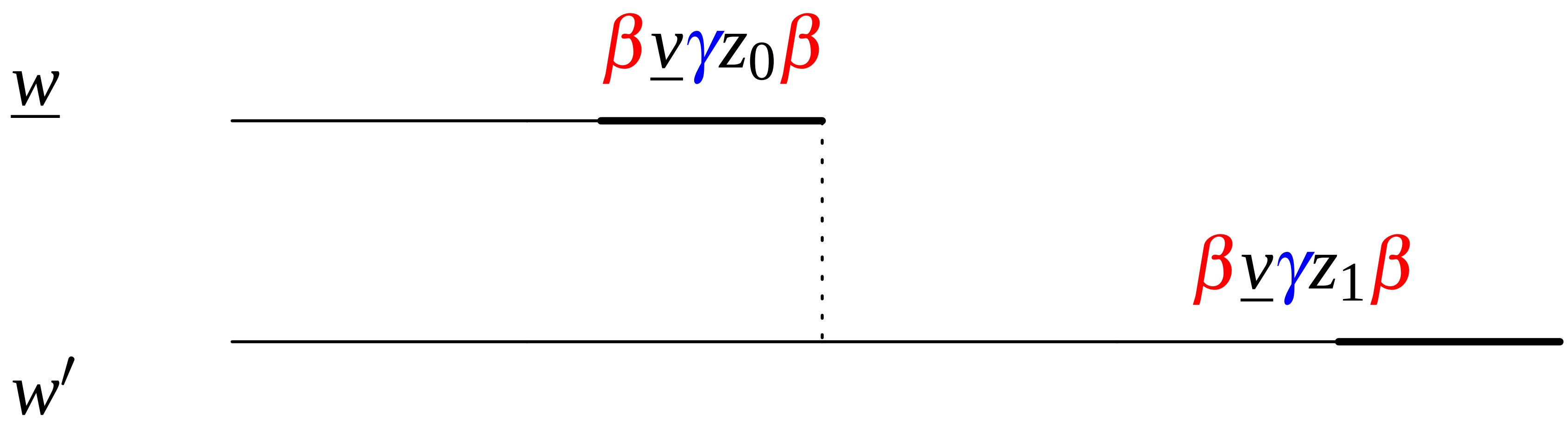

WTC **interprets** R

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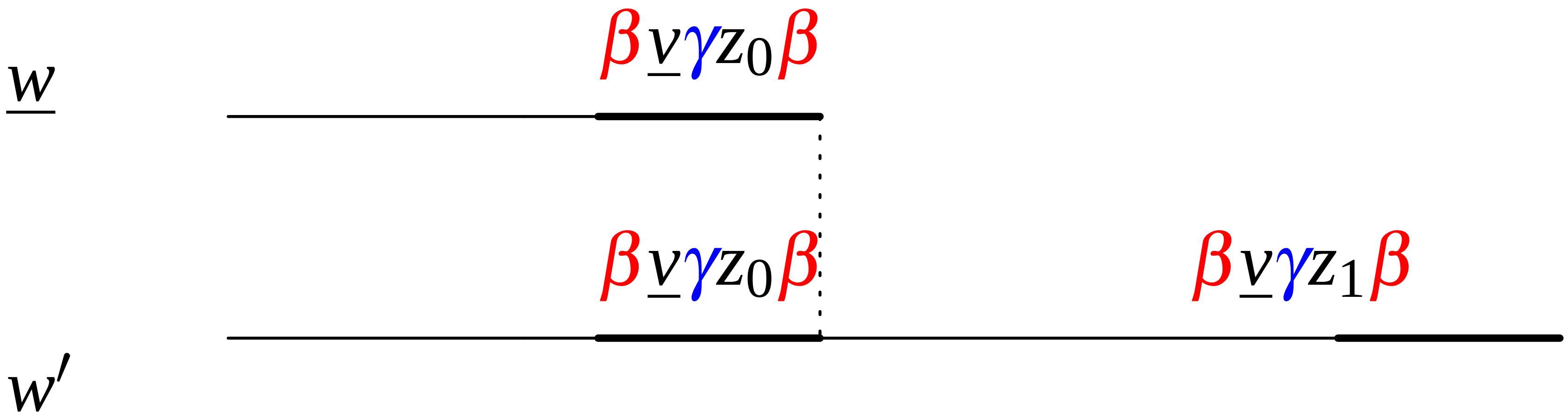

WTC interprets R

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