An Application of Turing Degree Theory to the ω-Decomposability Problem on Borel Functions

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Feb. 18, 2013

Computability Theory and Foundation of Mathematics 2013

Two Keywords

1. The Shore-Slaman Join Theorem (1999)
   - It was proved by using Kumabe-Slaman forcing.
   - It was used to show the first-order definability of the Turing jump in $\mathcal{D}_T$.
2. The Decomposability Problem of Borel Functions
   - The original decomposability problem was proposed by Luzin (191?) and negatively answered by Keldis (1934).
   - The modified decomposability problem was proposed by Andretta (2007), Semmes (2009), Pawlikowski-Sabok (2012), Motto Ros (201?).

Decomposing a hard function F into easy functions

Decomposing a discontinuous function F into easy functions

Decomposing a discontinuous function F into continuous functions

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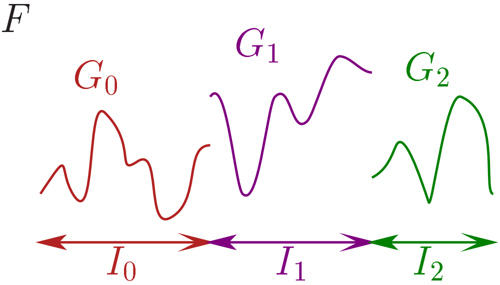
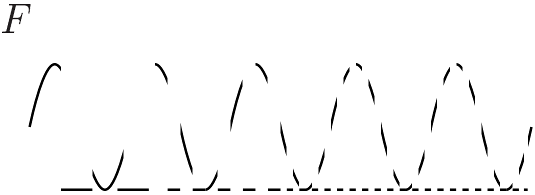


$$F(x) = \begin{cases} G_0(x) & \text{if } x \in I_0 \\ G_1(x) & \text{if } x \in I_1 \\ G_2(x) & \text{if } x \in I_2 \end{cases}$$

Decomposing a discontinuous function into continuous functions



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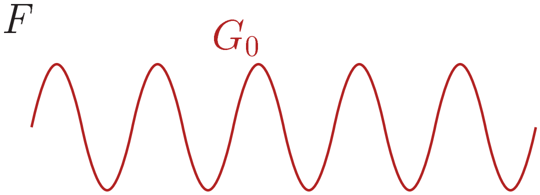

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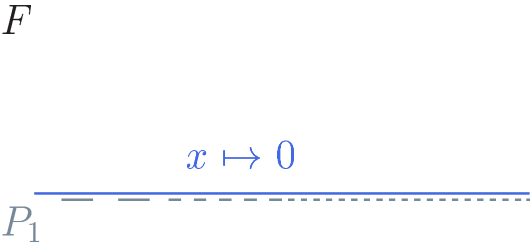

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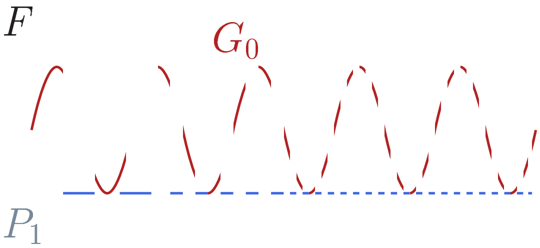

$$F(x) = \begin{cases} G_0(x) & \text{if } x \notin P_1 \\ 0 & \text{if } x \in P_1 \end{cases}$$

Decomposing a discontinuous function into continuous functions

$$\mathrm{Dirichlet}(x) = \lim_{m\to\infty} \lim_{n\to\infty} \cos^{2n}(m!\pi x)$$

$$\Downarrow$$

$$\mathrm{Dirichlet}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}. \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

If F is a Borel measurable function on $\mathbb{R}$, then can it be presented by using a countable partition $\{P_n\}_{n\in\omega}$ of $\mathbf{dom}(F)$ and a countable list $\{G_n\}_{n\in\omega}$ of continuous functions as follows?

$$F(x) = \begin{cases} G_0(x) & \text{if } x \in P_0 \\ G_1(x) & \text{if } x \in P_1 \\ G_2(x) & \text{if } x \in P_2 \\ G_3(x) & \text{if } x \in P_3 \\ \vdots & \vdots \end{cases}$$

Luzin's Problem (almost 100 years ago)

Can every Borel function on $\mathbb{R}$ be decomposed into countably many continuous functions?

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Example

The Turing jump $\boldsymbol{TJ} : \mathbf{2}^{\omega} \rightarrow \mathbf{2}^{\omega}$ is $\mathbf{\Sigma}^0_2$-measurable,
but it is indecomposable!

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Example

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Question

Which Borel function is decomposable into countably many continuous functions?

$$\text{Borel} = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$$

Definition

1. A function $F : X \to Y$ is Borel if
$$A \in \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(Y) \implies F^{-1}[A] \in \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X).$$
2. A function $F : X \to Y$ is $\mathbf{\Sigma}^0_\alpha$-measurable if
$$A \in \mathbf{\Sigma}^0_1(Y) \implies F^{-1}[A] \in \mathbf{\Sigma}^0_\alpha(X).$$
3. A function $F : X \to Y$ is $\mathbf{\Sigma}_{\alpha,\beta}$ if
$$A \in \mathbf{\Sigma}^0_\alpha(Y) \implies F^{-1}[A] \in \mathbf{\Sigma}^0_\beta(X).$$

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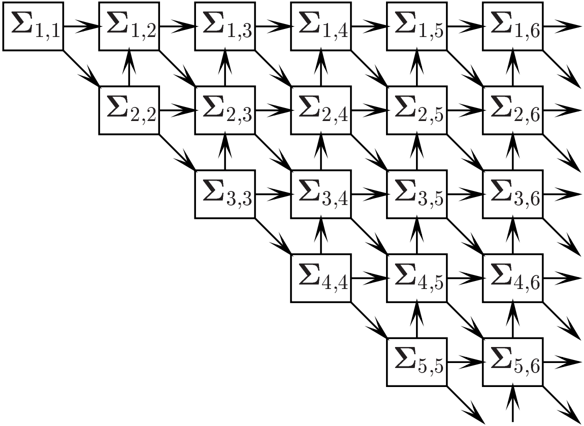

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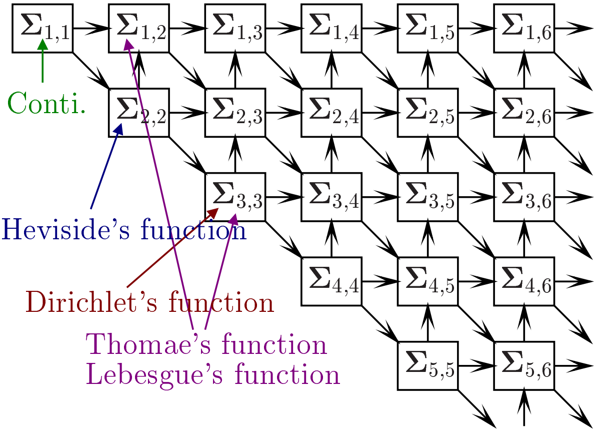

Definition

$F \in \mathbf{dec}(\boldsymbol{\Sigma}_\alpha)$ if it is decomposable into countably many $\boldsymbol{\Sigma}^0_\alpha$-measurable functions

- (Keldis 1934) $\boldsymbol{\Sigma}_{1,\alpha+1} \not\subseteq \mathbf{dec}(\boldsymbol{\Sigma}_\alpha)$
  i.e., there is a $\boldsymbol{\Sigma}^0_{\alpha+1}$-measurable function which is not decomposable into countably many $\boldsymbol{\Sigma}^0_\alpha$-measurable functions!
- The α-th Turing jump $x \mapsto x^{(\alpha)}$ is such a function.

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Problem

Given $(\alpha, \beta, \gamma) \in (\omega_1)^3$, determine whether or not

$$\boxed{\mathbf{\Sigma}_{\alpha,\beta}} \subseteq \boxed{\mathbf{dec}(\mathbf{\Sigma}_\gamma)}$$

Definition

F: a function from a top. sp. X into a top. sp. Y.

- $F \in \mathbf{dec}(\Sigma_\alpha)$ if it is decomposable into countably many Σ^0_α-measurable functions.
- $F \in \mathbf{dec}_\beta(\Sigma_\alpha)$ if it is decomposable into countably many Σ^0_α-measurable functions with Π^0_β domains,

  that is, there are a list $\{P_n\}_{n\in\omega}$ of Π^0_β subsets of X with $X = \bigcup_n P_n$ and a list $\{G_n\}_{n\in\omega}$ of Σ^0_α-measurable functions such that $F \upharpoonright P_n = G_n \upharpoonright P_n$ holds for all $n \in \omega$.

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The Jayne-Rogers Theorem 1982

X, Y: metric separable, X: analytic
For the class of all functions from X into Y,

$$\boxed{\Sigma_{2,2}} = \boxed{\mathbf{dec}_1(\Sigma_1)}$$

Borel Functions and Decomposability

	1	2	3	4	5	6
1	$\mathbf{\Sigma}_1$	$\mathbf{\Sigma}_2$	$\mathbf{\Sigma}_3$	$\mathbf{\Sigma}_4$	$\mathbf{\Sigma}_5$	$\mathbf{\Sigma}_6$
2	–	$\mathbf{dec}_1\mathbf{\Sigma}_1$	?	?	?	?
3	–	–	?	?	?	?
4	–	–	–	?	?	?
5	–	–	–	–	?	?
6	–	–	–	–	–	?

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Generalizing the Jayne-Rogers Theorem

- Gandy-Harrington topology (Solecki 1998)
- Generalization of Solecki dichotomy
  (Zapletal 2004; Motto Ros 2012; Pawlikowski-Sabok 2012)
- Infinite games and Wadge determinacy
  (Duparc 2001; Andretta 2007; Semmes 2009)

Theorem (Semmes 2009); determinacy + priority argument

For the class of functions on ω^ω,

$$\boxed{\mathbf{\Sigma}_{2,3}} = \boxed{\mathbf{dec}_2(\mathbf{\Sigma}_2)}$$

$$\boxed{\mathbf{\Sigma}_{3,3}} = \boxed{\mathbf{dec}_2(\mathbf{\Sigma}_1)}$$

The second level decomposability of Borel functions

	1	2	3	4	5	6
1	$\mathbf{\Sigma}_1$	$\mathbf{\Sigma}_2$	$\mathbf{\Sigma}_3$	$\mathbf{\Sigma}_4$	$\mathbf{\Sigma}_5$	$\mathbf{\Sigma}_6$
2	–	$\mathbf{dec}_1\mathbf{\Sigma}_1$	$\mathbf{dec}_2\mathbf{\Sigma}_2$	?	?	?
3	–	–	$\mathbf{dec}_2\mathbf{\Sigma}_1$	?	?	?
4	–	–	–	?	?	?
5	–	–	–	–	?	?
6	–	–	–	–	–	?

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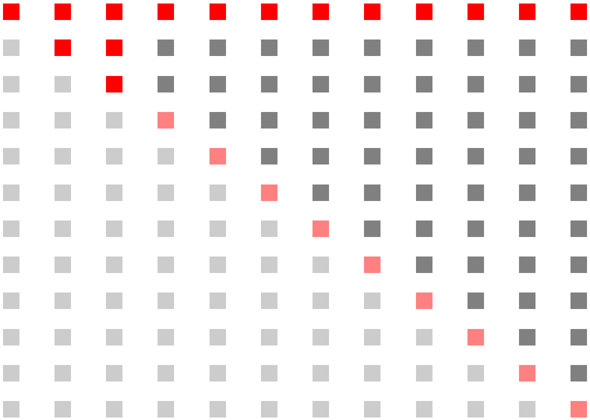
The Decomposability Problem

	1	2	3	4	5	6
1	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6
2	–	$\mathrm{dec}_1\Sigma_1$	$\mathrm{dec}_2\Sigma_2$	$\mathrm{dec}_3\Sigma_3$	$\mathrm{dec}_4\Sigma_4$	$\mathrm{dec}_5\Sigma_5$
3	–	–	$\mathrm{dec}_2\Sigma_1$	$\mathrm{dec}_3\Sigma_2$	$\mathrm{dec}_4\Sigma_3$	$\mathrm{dec}_5\Sigma_4$
4	–	–	–	$\mathrm{dec}_3\Sigma_1$	$\mathrm{dec}_4\Sigma_2$	$\mathrm{dec}_5\Sigma_3$
5	–	–	–	–	$\mathrm{dec}_4\Sigma_1$	$\mathrm{dec}_5\Sigma_2$
6	–	–	–	–	–	$\mathrm{dec}_5\Sigma_1$

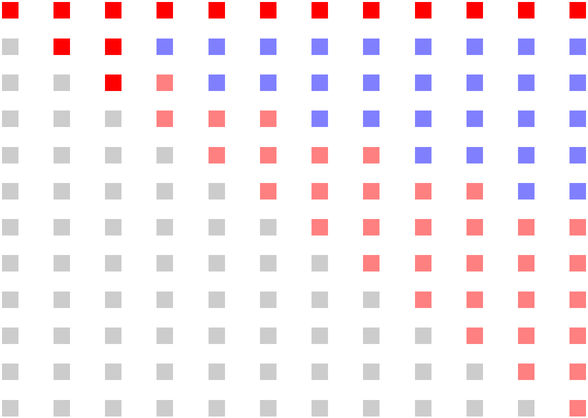
The Decomposability Conjecture (Andretta, Motto Ros et al.)

$$\boxed{\Sigma_{m+1,n+1}} = \boxed{\mathrm{dec}_n(\Sigma_{n-m+1})}$$

Overview of Previous Research



Main Theorem



Definition (de Brecht-Pauly 2012)

- F is $\mathbf{\Sigma}_{\alpha,\beta}$ iff $F^{-1}[\cdot] \upharpoonright \mathbf{\Sigma}^0_\alpha$ is a function from $\mathbf{\Sigma}^0_\alpha$ into $\mathbf{\Sigma}^0_\beta$.
- F is $\mathbf{\Sigma}^*_{\alpha,\beta}$ if $F^{-1}[\cdot] \upharpoonright \mathbf{\Sigma}^0_\alpha$ is continuous, as a function from $\mathbf{\Sigma}^0_\alpha$ into $\mathbf{\Sigma}^0_\beta$.
  Here the class of all $\mathbf{\Sigma}^0_\alpha$ subsets of a topological space is endowed with the quotient topology given by the canonical Borel code up to $\mathbf{\Sigma}^0_\alpha$.

Remark (Brattka 2005)

$\mathbf{\Sigma}_{1,\alpha} = \mathbf{\Sigma}^*_{1,\alpha}$ for every $\alpha < \omega_1$.

The Decomposability Problem

$$\boxed{\Sigma_{m+1,n+1}} = \boxed{\mathrm{dec}_n(\Sigma_{n-m+1})}$$

Main Theorem (K.)

For every $m, n \in \mathbb{N}$,

$$\boxed{\Sigma^*_{m+1,n+1}} \subseteq \boxed{\mathrm{dec}(\Sigma^0_{n-m+1})}$$

Moreover, if $2 \leq m \leq n < 2m$ then

$$\boxed{\Sigma^*_{m+1,n+1}} = \boxed{\mathrm{dec}_n(\Sigma^0_{n-m+1})}$$

The decomposability of continuously Borel functions

	1	2	3	4	5	6
1	$\mathbf{\Sigma}_1$	$\mathbf{\Sigma}_2$	$\mathbf{\Sigma}_3$	$\mathbf{\Sigma}_4$	$\mathbf{\Sigma}_5$	$\mathbf{\Sigma}_6$
2	–	$\mathbf{dec}_1\mathbf{\Sigma}_1$	$\mathbf{dec}_2\mathbf{\Sigma}_2$	?	?	?
3	–	–	$\mathbf{dec}_2\mathbf{\Sigma}_1$	$\mathbf{dec}_3\mathbf{\Sigma}_2$	?	?
4	–	–	–	$\mathbf{dec}_3\mathbf{\Sigma}_1$	$\mathbf{dec}_4\mathbf{\Sigma}_2$	$\mathbf{dec}_5\mathbf{\Sigma}_3$
5	–	–	–	–	$\mathbf{dec}_4\mathbf{\Sigma}_1$	$\mathbf{dec}_5\mathbf{\Sigma}_2$
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Sketch of Proof of $\mathbf{\Sigma}^*_{m+1,n+1} \subseteq \mathbf{dec}(\mathbf{\Sigma}_{n-m+1})$

Lemma (Lightface Analysis)

Let $F : 2^\omega \to 2^\omega$ be a function, and let p, q be oracles. Assume that the preimage $F^{-1}[A]$ of any lightface $\Sigma^{0,p}_m$ class A under F forms a lightface $\Delta^{0,p\oplus q}_{n+1}$ class, and one can effectively find an index of $F^{-1}[A]$ from an index of A.
Then $(F(x) \oplus p)^{(m)} \leq_T (x \oplus p \oplus q)^{(n)}$ for every $x \in 2^\omega$.

Lemma (Boldface)

$F \in \mathbf{\Sigma}^*_{m+1,n+1}$ iff the preimage of any $\mathbf{\Sigma}^0_m$ class under F forms a $\mathbf{\Delta}^0_{n+1}$ class.

Lemma (Boldface Analysis)

If $F \in \mathbf{\Sigma}^*_{m+1,n+1}$, then there exists $q \in 2^\omega$ such that $(F(x) \oplus p)^{(m)} \leq_T (x \oplus p \oplus q)^{(n)}$ for all $p \in 2^\omega$.

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Shore-Slaman Join Theorem 1999

The following sentence is true in the Turing degree structure.

$$(\forall \mathbf{a}, \mathbf{b})(\exists \mathbf{c} \geq \mathbf{a})[((\forall \zeta < \xi)\ \mathbf{b} \not\leq \mathbf{a}^{(\zeta)}) \rightarrow (\mathbf{c}^{(\xi)} \leq \mathbf{b} \oplus \mathbf{a}^{(\xi)} \leq \mathbf{b} \oplus \mathbf{c})$$

Lemma (Boldface Analysis; Restated)

If $F \in \mathbf{\Sigma}^*_{m+1,n+1}$, then there exists $q \in 2^\omega$ such that $(F(x) \oplus p)^{(m)} \leq_T (x \oplus p \oplus q)^{(n)}$ for all $p \in 2^\omega$.

Decomposition Lemma

$F \in \mathbf{\Sigma}^*_{m+1,n+1} \Rightarrow (\exists q)\ F(x) \leq_T (x \oplus q)^{(n-m)}$.

Sketch of Proof of $\mathbf{\Sigma}^*_{m+1,n+1} \subseteq \mathbf{dec}(\mathbf{\Sigma}_{n-m+1})$

Decomposition Lemma; Restated

$F \in \mathbf{\Sigma}^*_{m+1,n+1} \Rightarrow (\exists q)\ F(x) \leq_T (x \oplus q)^{(n-m)}$.

Corollary

$F \in \mathbf{\Sigma}^*_{m+1,n+1} \Rightarrow (\forall x)(\exists e)\ F(x) = \Phi_e((x \oplus q)^{(n-m)})$.

Sketch of Proof of $\Sigma^*_{m+1,n+1} \subseteq \mathbf{dec}(\Sigma_{n-m+1})$

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- $G_e : x \mapsto \Phi_e(x \oplus q)^{(n-m)}$ is Σ^0_{n-m+1}-measurable.
- $P_e := \{x \in \mathrm{dom}(G_e) : F(x) = G_e(x)\}$.

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- Then $F \upharpoonright P_e = G_e \upharpoonright P_e$, and $\mathbf{dom}(F) = \bigcup_e P_e$.

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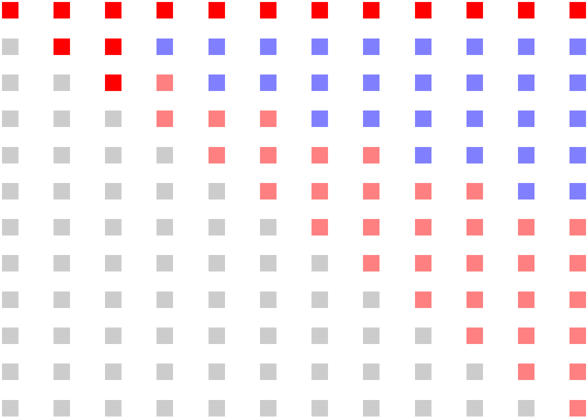
- $P_e := \{x \in \mathbf{dom}(G_e) : F(x) = G_e(x)\}$.
- Then $F \upharpoonright P_e = G_e \upharpoonright P_e$, and $\mathbf{dom}(F) = \bigcup_e P_e$.
- Consequently, $\mathbf{\Sigma}^*_{m+1,n+1} \subseteq \mathbf{dec}(\mathbf{\Sigma}_{n-m+1})$

Main Theorem



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Main Theorem (K.)

For every $m, n \in \mathbb{N}$,

$$\Sigma^*_{m+1,n+1} \subseteq \mathrm{dec}(\Sigma^0_{n-m+1})$$

Moreover, if $2 \leq m \leq n < 2m$ then

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