Computably measurable sets and computably measurable functions in terms of algorithmic randomness

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Motivation

- Measure (Probability) theory everywhere(!)
- Non-constructive proof

Topics in measure theory

- Measure
- Measurable set
- Measurable function
- Lebesgue integral
- Radon-Nikodym theorem
- Change of variables
- Fourier transform
- L^p spaces
- convergence of measure
- conditional measure

Use randomness

- A property holds
  almost surely (or almost everywhere)
- A property holds for a (sufficiently) random point

- differentiable
- Birkhoff's ergodic theorem

computably measurable set

approximation approach

$[0,1]$ with the Lebesgue measure μ

Sanin 1968(!), Edalat 2009, Hoyrup& Rojas 2009, Rute

$\mathcal{B}$: the set of Borel subsets

$$d(A,B) = \mu(A \Delta B)$$

$[\mathcal{B}]$: the quotient of $\mathcal{B}$ by $A \sim B \iff d(A,B) = 0$

$\mathcal{U}$: the set of finite unions of intervals with rational endpoints

Theorem (Rojas 2008)

$([\mathcal{B}], d, \mathcal{U})$ is a computable metric space

For a subset $A \subseteq [0,1]$, $[A] \in [\mathcal{B}]$ is a computable point in the space if there exists a computable sequence $\{B_n\}$ of U such that $d(A, B_n) \leq 2^{-n}$ for all n.

Naive definition

A is a computably measurable set if $[A]$ is a computable point in the space.

Remark

Essentially the same idea is used in Pour-El & Richard (1989).

The relation with randomness

- Sanin or Edalat didn't study
- Hoyrup-Rojas did for Martin-Löf randomness
- Rute did for Schnorr randomness but not fully effective

Convergence

Observation (Implicit in Pathak et al., Rute and M.)

The following are equivalent for $x \in [0,1]$:

1. x is Schnorr random,
2. $\lim_n B_n(x)$ exits for each computable sequence $\{B_n\}$ in $\mathcal{U}$ such that

$$d(B_{n+1}, B_n) \leq 2^{-n}$$

for all n.

Possible definition

Let $\{A_n\}$ be a computable sequence of $\mathcal{U}$ such that

$$d(A_{n+1}, A_n) \leq 2^{-n}$$

for all n. The set A defined by

$$A(x) = \begin{cases} \lim_n A_n(x) & \text{if } x \text{ is Schnorr random} \\ 0 & \text{otherwise.} \end{cases}$$

is called a computably measurable set.

This idea is similar to $\hat{f}$ in Pathak et al. and Rute.

Definition

Definition (M.)

A set A is called a computably measurable set if there is a computable sequence $\{A_n\}$ of $\mathcal{U}$ such that $d(A_{n+1}, A_n) \leq 2^{-n}$ for all n and $A(x)$ is equivalent to $\lim_n A_n(x)$ up to Schnorr null.

Schnorr null

An open U is c.e. if $U = \bigcup_n U_n$ for a computable $\{U_n\}$ in $\mathcal{U}$.

Definition (Schnorr 1971)
A Schnorr test is a sequence $\{U_n\}$ of uniformly c.e. open sets with $\mu(U_n) \leq 2^{-n}$ for each n. A point x is called Schnorr random if $x \notin \bigcap_n U_n$ for each Schnorr test.

For each Schnorr test $\{U_n\}$, the set $\bigcap_n U_n$ is called a Schnorr null set.

No universal Schnorr test

Proposition

For each Schnorr null set N, there is a computable point z that is not contained in N.

Definition

A and B are equivalent up to Schnorr null if $A \Delta B$ is contained in a Schnorr null set.

Remark

Equivalence up to Schnorr null is a stronger notion than equivalence for all random points.

Definition (again)

Definition (M.)

A set A is called a computably measurable set if there is a computable sequence $\{A_n\}$ of $\mathcal{U}$ such that $d(A_{n+1}, A_n) \leq 2^{-n}$ for all n and $A(x)$ is equivalent to $\lim_n A_n(x)$ up to Schnorr null.

Possible definition

Let $\{A_n\}$ be a computable sequence of $\mathcal{U}$ such that

$$d(A_{n+1}, A_n) \leq 2^{-n}$$

for all n. The set A defined by

$$A(x) = \begin{cases} \lim_n A_n(x) & \text{if } x \text{ is Schnorr random} \\ 0 & \text{otherwise.} \end{cases}$$

is called a computably measurable set.

This idea is similar to $\hat{f}$ in Pathak et al. and Rute.

Usual definition

A is computably measurable set if $[A]$ is a computable point, that is, there is a computable sequence $\{A_n\}$ of $\mathcal{U}$ such that

$$d(A, A_n) = \mu(A \Delta A_n) \leq 2^{-n}.$$

Sometimes called effectively measurable set or μ-recursive sets

Basic property

Proposition

Every computable measurable set has a computable measure.

Proposition

Let A, B be computable measurable sets.

Then so are A^c, $A \cup B$ and $A \cap B$.

Furthermore, $\mu(A \Delta B) = 0$ iff A and B are equivalent up to Schnorr null.

The approach via regularity

This approach is used in Edalat and Hoyrup & Rojas.

Proposition The following are equivalent for a set A:

(i) A is a computably measurable set A.

(ii) There are two sequences $\{U_n\}$ and $\{V_n\}$ of c.e. open sets such that

$$V_n^c \subseteq A \subseteq U_n,$$

$\mu(U_n \cap V_n) \leq 2^{-n}$ and $\mu(U_n \cap V_n)$ is uniformly computable for each n.

Proposition

Let $E \subseteq \mathbb{R}$ be a measurable set.

(i) For any $\epsilon > 0$, there is an open set $O \supseteq E$ such that $m(O \setminus E) < \epsilon$.

(ii) For any $\epsilon > 0$, there is a closed set $F \subseteq E$ such that $m(E \setminus F) < \epsilon$.

(iii) There is a $G \in G_\delta$ such that $E \subseteq G$ and $m(G \setminus E) = 0$.

(iv) There is a $F \in F_\sigma$ such that $E \supseteq F$ and $m(E \setminus F) = 0$.

Furthermore, if $m(E) < \infty$, then, for any $\epsilon > 0$, there is a finite union U of open intervals such that $m(U \Delta E) < \epsilon$.

Proposition

The following are equivalent for a set A:

(i) A is a computably measurable set A.

(ii) A has a computable measure and is equivalent up to Schnorr null to $\bigcap_n U_n$ for a decreasing sequence $\{U_n\}$ of uniformly c.e. open sets such that $\mu(U_n)$ is uniformly computable.

Definition (M.)

A function $f :\subseteq X \to Y$ is Schnorr layerwise computable if there exists a Schnorr test $\{U_n\}$ such that

$$f|_{X \setminus U_n}$$

is uniformly computable.

Proposition

The following are equivalent for a set A:

(i) A is a computably measurable set,

(ii) $A : [0,1] \to \{0,1\}$ is Schnorr layerwise computable.

computably measurable function

Definition

A function $f : X \to Y$ is measurable if $f^{-1}(U)$ is measurable for each open set U.

Theorem (Lusin's theorem)

A function $f : [0,1] \to \mathbb{R}$ is measurable iff, for each $\epsilon > 0$, there is a continuous function f_ϵ and a compact set K_ϵ such that $\mu(K_\epsilon^c) < \epsilon$ and $f = f_\epsilon$ on K_ϵ.

Definition (M.)

A function $f : [0,1] \to \mathbb{R}$ is computably measurable if $f^{-1}(U)$ is uniformly computably measurable for each interval U with rational endpoints.

Theorem (M.)

A function $f : [0,1] \to \mathbb{R}$ is computably measurable iff Schnorr layerwise computable.

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