A short proof of two Ramsey like independence results using recursion theoretic methods

Florian Pelupessy[1]

Department of Mathematics, Ghent University, Belgium

Tokyo, 18 February, 2013

UNIVERSITEIT GENT

FACULTY OF SCIENCES

[1] Joint work with Harvey Friedman

Overview

1. Introduction: two Ramsey like theorems which are independent of PA
2. Preliminaries
3. Proofs

Introduction

Two Ramsey like theorems

Theorem (Paris–Harrington, 1977)

For every d, c, m there exists an R such that for every colouring $C\colon [m,R]^d \to c$ there exists an $H \subseteq [m,R]$ of size $\min H$ for which C restricted to $[H]^d$ is constant.

Two Ramsey like theorems

$(a_1, \ldots, a_r) \leq (b_1, \ldots, b_r) \Leftrightarrow a_1 \leq b_1 \wedge \cdots \wedge a_r \leq b_r$

We call a function $C\colon \{0, \ldots, R\}^d \to \mathbb{N}^r$ limited if $\max C(x) \leq \max x$.

Theorem (Adjacent Ramsey, Friedman, 2010)

For every d, r there exists R such that for every limited function $C\colon \{0, \ldots, R\}^d \to \mathbb{N}^r$ there are $x_1 < \cdots < x_{d+1} \leq R$ with $C(x_1, \ldots, x_d) \leq C(x_2, \ldots, x_{d+1})$.

Preliminaries

Ordinals

$$0, 1, 2, 3, 4, 5, 6, \ldots, \omega,$$
$$\omega + 1, \omega + 2, \omega + 3, \ldots, \omega + \omega = \omega \cdot 2, \ldots, \omega \cdot \omega = \omega^2,$$
$$\omega^2 + 1, \ldots, \omega^\omega = \omega_2, \ldots, \omega^{\omega_2} = \omega_3, \ldots, \omega_\omega = \varepsilon_0.$$

Ordinals: Cantor Normal Forms

All $\alpha < \varepsilon_0$ can be written uniquely in the Cantor Normal Form:

$$\alpha = \omega^{\alpha_1} \cdot m_1 + \cdots + \omega^{\alpha_n} \cdot m_n,$$

where $\alpha_1 > \cdots > \alpha_n$ and $m_1 > 0, \ldots, m_n > 0$, $n \geq 1$.

Ordinals: Fundamental sequences

$$
\begin{aligned}
(\alpha+1)[x] &= \alpha, \\
(\alpha+\omega^{\alpha_n+1}\cdot(m+1))[x] &= \alpha+\omega^{\alpha_n+1}\cdot m+\omega^{\alpha_n}\cdot x, \\
(\alpha+\omega^{\gamma}\cdot(m+1))[x] &= \alpha+\omega^{\gamma}\cdot m+\omega^{\gamma[x]}.
\end{aligned}
$$

Ordinals: Hydra battles

A Hydra battle is a sequence $\omega_d = h_0 > h_1 > \dots$ of ordinals such that $h_{i+1} = h_i[i+1]$.

Termination of Hydra battles is known to be independent of PA.

Ordinals: Some definitions

Given $\alpha = \omega^{\alpha_1} \cdot a_1 + \cdots + \omega^{\alpha_n} \cdot a_n$ and $\beta = \omega^{\beta_1} \cdot b_1 + \cdots + \omega^{\beta_m} \cdot b_m$.

1. $\mathrm{CP}(\alpha, \beta)$ is the smallest i such that $\omega^{\alpha_i} \cdot a_i \neq \omega^{\beta_i} \cdot b_i$ if such an i exists, zero otherwise.
2. $\mathrm{CC}(\alpha, \beta)$ is $a_{\mathrm{CP}(\alpha,\beta)}$, where $a_0 = 0$.
3. $\mathrm{CE}(\alpha, \beta)$ is $\alpha_{\mathrm{CP}(\alpha,\beta)}$, where $\alpha_0 = 0$.
4. $\mathrm{MP}(\alpha) = \max\{n, \mathrm{MP}(\alpha_i)\}$.
5. $\mathrm{MC}(\alpha) = \max\{a_i, \mathrm{MC}(\alpha_i)\}$.

Ordinals: Some definitions

1. $F_1(\alpha) = \alpha$.
2. $F_{d+1}(\alpha_1, \ldots, \alpha_{d+1}) =$
   $(\mathrm{CP}(\alpha_1, \alpha_2), \mathrm{CC}(\alpha_1, \alpha_2), F_d(\mathrm{CE}(\alpha_1, \alpha_2), \ldots, \mathrm{CE}(\alpha_d, \alpha_{d+1})))$.

Some easy lemmas

Lemma

$\mathrm{CP}(\alpha, \beta) \leq \mathrm{CP}(\beta, \gamma) \wedge \mathrm{CE}(\alpha, \beta) \leq \mathrm{CE}(\beta, \gamma) \wedge \mathrm{CC}(\alpha, \beta) \leq \mathrm{CC}(\beta, \gamma) \Rightarrow \alpha \leq \beta$.

Lemma

$F_d(\alpha_1, \ldots, \alpha_d) \leq F_d(\alpha_2, \ldots, \alpha_{d+1}) \Rightarrow \alpha_1 \leq \alpha_2$.

Some easy lemmas

Lemma

$\mathrm{CP}(\alpha, \beta) \leq \mathrm{CP}(\beta, \gamma) \wedge \mathrm{CE}(\alpha, \beta) \leq \mathrm{CE}(\beta, \gamma) \wedge \mathrm{CC}(\alpha, \beta) \leq \mathrm{CC}(\beta, \gamma) \Rightarrow \alpha \leq \beta$.

Lemma

$F_d(\alpha_1, \ldots, \alpha_d) \leq F_d(\alpha_2, \ldots, \alpha_{d+1}) \Rightarrow \alpha_1 \leq \alpha_2$.

Lemma

$\max F_d(\alpha_1, \ldots, \alpha_d) \leq \mathrm{MP}(\alpha_1), \mathrm{MC}(\alpha_1)$.

Lemma

For every Hydra battle $h_0, h_1, \ldots$ we have $\mathrm{MP}(h_i), \mathrm{MC}(h_i) \leq i$.

Proofs

AR is independent

$$C(x_1, \ldots, x_d) = F(h_{x_1}, \ldots, h_{x_d})$$

AR is independent

$$C(x_1, \ldots, x_d) = F(h_{x_1}, \ldots, h_{x_d})$$

$$C(x_1, \ldots, x_d) \leq C(x_2, \ldots, x_{d+1}) \Rightarrow h_{x_1} \leq h_{x_2}$$

AR is independent

$$C(x_1, \ldots, x_d) = F(h_{x_1}, \ldots, h_{x_d})$$

$$C(x_1, \ldots, x_d) \leq C(x_2, \ldots, x_{d+1}) \Rightarrow h_{x_1} \leq h_{x_2}$$

Hence AR implies that every Hydra battle terminates.

PH is independent

$$C(x_1, \ldots, x_{d+1}) = \begin{cases} 0 & \text{if } F_d(h_{x_1-d-2}, \ldots, h_{x_d-d-2}) \leq \\ & \quad F_d(h_{x_2-d-2}, \ldots, h_{x_{d+1}-d-2}), \\ i & \text{otherwise,} \end{cases}$$

PH is independent

$$C(x_1,\dots,x_{d+1}) = \begin{cases} 0 & \text{if } F_d(h_{x_1-d-2},\dots,h_{x_d-d-2}) \leq \\ & \quad F_d(h_{x_2-d-2},\dots,h_{x_{d+1}-d-2}), \\ i & \text{otherwise,} \end{cases}$$

where i is the least such that:

$$(F_d(h_{x_1-d-2},\dots,h_{x_d-d-2}))_i > (F_d(h_{x_2-d-2},\dots,h_{x_{d+1}-d-2}))_i.$$

PH is independent

$$C(x_1,\ldots,x_{d+1}) = \begin{cases} 0 & \text{if } F_d(h_{x_1-d-2},\ldots,h_{x_d-d-2}) \leq \\ & \quad F_d(h_{x_2-d-2},\ldots,h_{x_{d+1}-d-2}), \\ i & \text{otherwise,} \end{cases}$$

where i is the least such that:

$$(F_d(h_{x_1-d-2},\ldots,h_{x_d-d-2}))_i > (F_d(h_{x_2-d-2},\ldots,h_{x_{d+1}-d-2}))_i.$$

If $C(x_1,\ldots) = C(x_2,\ldots) = \cdots = C(x_{x_1-d},\ldots) \neq 0$ then we obtain a sequence $x_1 - d - 2 \geq i_1 > \cdots > i_{x_1-d}$ which is impossible.

PH is independent

$$C(x_1, \dots, x_{d+1}) = \begin{cases} 0 & \text{if } F_d(h_{x_1-d-2}, \dots, h_{x_d-d-2}) \leq \\ & \quad F_d(h_{x_2-d-2}, \dots, h_{x_{d+1}-d-2}), \\ i & \text{otherwise,} \end{cases}$$

where i is the least such that:

$$(F_d(h_{x_1-d-2}, \dots, h_{x_d-d-2}))_i > (F_d(h_{x_2-d-2}, \dots, h_{x_{d+1}-d-2}))_i.$$

If $C(x_1, \dots) = C(x_2, \dots) = \dots = C(x_{x_1-d}, \dots) \neq 0$ then we obtain a sequence $x_1 - d - 2 \geq i_1 > \dots > i_{x_1-d}$ which is impossible.

Hence PH implies that every Hydra battle terminates.

Concluding remark

These proofs can be modified to show that for $d > 1$:

adjacent Ramsey with fixed dimension d and

Paris–Harrington with fixed dimension $d + 1$

are independent of $\mathrm{I}\Sigma_d$.

Thank you for listening.

Florian Pelupessy
pelupessy@cage.ugent.be
cage.ugent.be/∼pelupessy/