On the Notions of Relative Randomness

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Computability Theory and Foundations of Mathematics

Outline

1. Questions

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Question 1

Can we define some randomness notions in terms of another randomness notions?

Question 2

How can we define it?

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Liang Yu: **Characterizing strong randomness via Martin-Löf randomness.** *Annals of Pure and Applied Logic, vol. 163, no. 3, pp. 214-224 (2012).*

Question 1

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Question 2

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Question 2

How can we define it?

Let R and S be two randomness notions.

Questoin 1'

$(\exists \Gamma \subset 2^{\omega})[\mathrm{R} = \bigcap_{X \in \Gamma} X\text{-}\mathrm{S}]$ or $(\exists \Gamma \subset 2^{\omega})[\mathrm{R} = \bigcup_{X \in \Gamma} X\text{-}\mathrm{S}]$?

where X-R and X-S are relativizations of R and S to X, respectively.

Question 2'

What kinds of Γ satisfy above relations ?

2. Randomness Notions

ML-randomness

ML-randomness is a central notion of algorithmic randomness for subsets of $\mathbb{N}$, which defined in the following way.

Definition (Martin-Löf , 1966)

(i) A Martin-Löf test, or ML-test for short, is a uniformly c.e. sequence $(G_m)_{m\in\mathbb{N}}$ of open sets such that $\forall m \in \mathbb{N}\ \mu(G_m) \leq 2^{-m}$.

(ii) A set $Z \subseteq \mathbb{N}$ fails the test if $Z \in \bigcap_m G_m$, otherwise Z passes the test.

(iii) Z is ML-random if Z passes each ML-test.

Let $MLR = \{X \mid X \text{ is ML-random }\}$.
Let Z-MLR=$\{X \mid X$ is ML-random relative to Z $\}$

Weak 2-randomness

Weak 2-randomness, like ML-randomness, is defined in terms of tests.

Definition (Kurtz, 1981)

(i) A generalized ML-test is a uniformly c.e. sequence $(G_m)_{m\in\mathbb{N}}$ of open sets such that $\mu(\bigcap_m G_m) = 0$.

(ii) Z is weakly 2-random if it passes every generalized ML-test.

Let $W2R = \{X \mid X \text{ is weakly 2-random }\}$.

Schnorr randomness

Definition (Schnorr, 1971)

A Schnorr test is a ML-test $(G_m)_{m\in\mathbb{N}}$ such that μG_m is computable uniformly in m. A set $Z \subseteq \mathbb{N}$ fails the test if $Z \in \bigcap_m G_m$, otherwise Z passes the test. Z is Schnorr random if Z passes each Schnorr test.

Let $SR = \{X \mid X \text{ is Schnorr-random}\}$.

Martingale

Another important notion of randomness is computable randomness, whose definition involves the concept of a martingle.

Definition

A martingale is a function $d : 2^{<\mathbb{N}} \to \mathbb{R}_{\geq 0}$ that satisfies for every $\sigma \in 2^{<\mathbb{N}}$ the averaging condition $d(\sigma) = \frac{d(\sigma 0)+d(\sigma 1)}{2}$.

A martingale d succeeds on a set A if $\limsup_{n\to\infty} d(A \upharpoonright n) = \infty$.

Definition

We say that Z is computably random if no computable martingale succeeds on Z.

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weak 2-randomness

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ML-randomness

↓

computably randomness

↓

Schnorr randomness

↓

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Lowness and Higness

Definition

Let R and S be two randomness notions. We identify these notions with the sets of all random reals in the sense of these notions.

$$\mathrm{Low}(\mathrm{R}, \mathrm{S}) = \{X \in 2^\omega : \mathrm{R} \subset X\text{-}\mathrm{S}\}$$

$$\mathrm{High}(\mathrm{R}, \mathrm{S}) = \{X \in 2^\omega : X\text{-}\mathrm{R} \subset \mathrm{S}\}$$

where X-R and X-S are relativizations of R and S to X, respectively.

Remark

Questoin 1'

$(\exists \Gamma \subset 2^\omega)[\mathrm{R} = \bigcap_{X \in \Gamma} X\text{-}\mathrm{S}]$ or $(\exists \Gamma \subset 2^\omega)[\mathrm{R} = \bigcup_{X \in \Gamma} X\text{-}\mathrm{S}]$?

We can prove easily Q1' is equivalent to Q1":

Questoin 1"

$\mathrm{R} = \bigcap_{X \in \mathrm{Low(R,S)}} X\text{-}\mathrm{S}$ or $\bigcup_{X \in \mathrm{High(R,S)}} X\text{-}\mathrm{R} = \mathrm{S}$?

This is because that: $R \subset \bigcap_{X \in low(R,S)} X - S \subset \bigcap_{X \in \Gamma} X - S$ for any $\Gamma \subset low(R,S)$. And if Γ satisfies the first equality of Q1', then $\Gamma \subset low(R,S)$.

3. Answers to Questions

Positive Answer: $\emptyset'$-SR vs MLR

Questoin 1'

$(\exists \Gamma \subset 2^\omega)[\mathrm{R} = \bigcap_{X \in \Gamma} X\text{-}\mathrm{S}]$ or $(\exists \Gamma \subset 2^\omega)[\mathrm{R} = \bigcup_{X \in \Gamma} X\text{-}\mathrm{S}]$?

Question 2'

What kinds of Γ satisfy above relations ?

Positive Answer: $\emptyset'$-SR vs MLR

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$(\exists\Gamma \subset 2^\omega)[\mathrm{R} = \bigcap_{X\in\Gamma} X\text{-}\mathrm{S}]$ or $(\exists\Gamma \subset 2^\omega)[\mathrm{R} = \bigcup_{X\in\Gamma} X\text{-}\mathrm{S}]$?

Question 2'

What kinds of Γ satisfy above relations ?

Theorem (Yu, 2012)

$\emptyset'$-Schnorr randomness $= \bigcap_{X\in\mathbb{L}} X - MLR$.

where $\mathbb{L}$ is the set of all the low sets.

Positive Answer: $\emptyset'$-SR vs MLR

Questoin 1'

$(\exists\Gamma \subset 2^\omega)[\mathrm{R} = \bigcap_{X\in\Gamma} X\text{-}\mathrm{S}]$ or $(\exists\Gamma \subset 2^\omega)[\mathrm{R} = \bigcup_{X\in\Gamma} X\text{-}\mathrm{S}]$?

Question 2'

What kinds of Γ satisfy above relations ?

Theorem (Yu, 2012)

$\emptyset'$*-Schnorr randomness* $= \bigcap_{X\in\mathbb{L}} X - MLR$.

where $\mathbb{L}$ is the set of all the low sets.

Question, (Yu, 2012)

Does $\emptyset'$-Schnorr randomness $= \bigcap_{X\in\mathbb{L}\cap\mathbb{G}} X - MLR$?

where $\mathbb{G}$ is the set of all the 1-generic sets.

Positive Answer: $\emptyset'$-SR vs MLR

Theorem

For any $\emptyset'$-Schnorr test $\{\mathcal{U}_e\}_{e\in\omega}$, there exist a low 1-generic real Z and a Z-Martin-Löf test $\{\mathcal{V}_e\}_{e\in\omega}$ with $\bigcap_{e\in\omega}\mathcal{U}_e \subset \bigcap_{e\in\omega}\mathcal{V}_e$.

Proof.

A finite injury argument. □

Corollary

$\emptyset'$-Schnorr randomness $= \bigcap_{X\in\mathbb{L}\cap\mathbb{G}} X - MLR$.

This give an affirmative answer to Yu's problem.

Application

Recall that a real A is said to be *LR-reducible* to B, abbreviated $A \leq_{\mathrm{LR}} B$, if every real Martin-Löf random relative to B is also Martin-Löf random relative to A.

Theorem (Diamondstone, 2012)

For any low real X, Y, there exists a low c.e. real Z such that $X, Y \leq_{\mathrm{LR}} Z$.

Application

Recall that a real A is said to be *LR-reducible* to B, abbreviated $A \leq_{\mathrm{LR}} B$, if every real Martin-Löf random relative to B is also Martin-Löf random relative to A.

Theorem (Diamondstone, 2012)

For any low real X, Y, there exists a low c.e. real Z such that $X, Y \leq_{\mathrm{LR}} Z$.

We have the following similar theorem:

Theorem

For any low real X, Y, there exists a low 1-generic real Z such that $X, Y \leq_{\mathrm{LR}} Z$.

Positive Answer: $\emptyset'$-SR vs MLR

Questoin 1'

$(\exists \Gamma \subset 2^\omega)[\mathrm{R} = \bigcap_{X \in \Gamma} X\text{-}\mathrm{S}]$ or $(\exists \Gamma \subset 2^\omega)[\mathrm{R} = \bigcup_{X \in \Gamma} X\text{-}\mathrm{S}]$?

Theorem (Yu, 2012)

$\bigcup_{X \in \mathrm{High}(\mathrm{MLR}, \emptyset'\text{-}\mathrm{SR})} X\text{-}\mathrm{MLR} = \emptyset'\text{-}\mathrm{SR}$

This is a positive answer for Q1' in the uniou part.
In fact, Yu also shown that Γ can be $MLR \cap High(ML, \emptyset' - SR)$. This is a interesting answer of Q2.

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What kinds of Γ satisfy above relations ?

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A New Characterization of MLR.

Theorem

$\bigcup_{X\in PA} X\text{-}\mathrm{CR} = \mathrm{MLR}.$

Proof.

Franklin, Stephan and Yu (2011) proved that $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$ includes all PA-complete reals. Reimann and Slaman showed that any Martin-Löf random is Martin-Löf relative to some PA-complete real(Randomness preservation basis theorem). Since $X\text{-}\mathrm{MLR} \subset X\text{-}\mathrm{CR}$, it is known that any Martin-Löf random is computably random relative to some PA-complete real. This implies the desired equality. □

Negative Answer: W2R vs MLR

Questoin 1'

$(\exists \Gamma \subset 2^{\omega})[\mathrm{R} = \bigcap_{X \in \Gamma} X\text{-}\mathrm{S}]$ or $(\exists \Gamma \subset 2^{\omega})[\mathrm{R} = \bigcup_{X \in \Gamma} X\text{-}\mathrm{S}]$?

Negative Answer: W2R vs MLR

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$(\exists\Gamma \subset 2^\omega)[\mathrm{R} = \bigcap_{X\in\Gamma} X\text{-}\mathrm{S}]$ or $(\exists\Gamma \subset 2^\omega)[\mathrm{R} = \bigcup_{X\in\Gamma} X\text{-}\mathrm{S}]$?

Theorem (Yu, 2012)

$\neg\exists\Gamma \subset 2^\omega$ *such that* $W2R = \bigcap_{x\in\Gamma} X - MLR$.

Negative Answer: W2R vs MLR

Questoin 1'

$(\exists \Gamma \subset 2^{\omega})[\mathrm{R} = \bigcap_{X \in \Gamma} X\text{-}\mathrm{S}]$ or $(\exists \Gamma \subset 2^{\omega})[\mathrm{R} = \bigcup_{X \in \Gamma} X\text{-}\mathrm{S}]$?

Theorem (Yu, 2012)

$\neg \exists \Gamma \subset 2^{\omega}$ *such that* $W2R = \bigcap_{x \in \Gamma} X - MLR.$

Theorem (Merkle and Yu, unpublished)

$\neg \exists \Gamma \subset 2^{\omega}$ *such that* $W2R = \bigcup_{x \in \Gamma} X - MLR.$

Negative Answer: SR vs CR

Questoin 1'

$(\exists \Gamma \subset 2^{\omega})[\mathrm{R} = \bigcap_{X \in \Gamma} X\text{-}\mathrm{S}]$ or $(\exists \Gamma \subset 2^{\omega})[\mathrm{R} = \bigcup_{X \in \Gamma} X\text{-}\mathrm{S}]$?

Negative Answer: SR vs CR

Questoin 1'

$(\exists \Gamma \subset 2^\omega)[\mathrm{R} = \bigcap_{X \in \Gamma} X\text{-}\mathrm{S}]$ or $(\exists \Gamma \subset 2^\omega)[\mathrm{R} = \bigcup_{X \in \Gamma} X\text{-}\mathrm{S}]$?

Theorem

$\mathrm{SR} = \bigcap_{X \in \mathrm{Low(CR,SR)}} X\text{-}\mathrm{SR} \neq \mathrm{CR}$

Proof.

Kjos-Hanssen, Nies and Stephan (2006) proved that $\mathrm{Low(CR, SR)} = \mathrm{Low(SR, SR)}$ holds. □

Theorem

$\emptyset' - SR = \bigcup_{X \in \mathrm{High(SR,CR)}} X\text{-}\mathrm{SR} \neq \mathrm{CR}$

Summary of Results

$$\bigcap_{X \in \mathrm{Low(R,S)}} X\text{-S} = \mathrm{R}$$

R \ S	0'-SR	W2R	MLR	CR	SR
0'-SR		Yes	Yes	?	Yes
W2R	?		No	No	No
MLR	Yes	No		?	?
CR	?	?	Yes		No
SR	Yes	No	No	No	

$$\bigcup_{X \in \mathrm{High(R,S)}} X\text{-R} = \mathrm{S}$$

Summary of Results

$$\bigcap_{X \in \mathrm{Low(R,S)}} X\text{-S} = \mathrm{R}$$

R \ S	0'-SR	W2R	MLR	CR	SR
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W2R	?		No	No	No
MLR	Yes	No		?	?
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SR	Yes	No	No	No	

$$\bigcup_{X \in \mathrm{High(R,S)}} X\text{-R} = \mathrm{S}$$

Summary of Results

$$\bigcap_{X \in \mathrm{Low(R,S)}} X\text{-S} = \mathrm{R}$$

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W2R	?		No	No	No
MLR	Yes	No		?	?
CR	?	?	Yes		No
SR	Yes	No	No	No	

$$\bigcup_{X \in \mathrm{High(R,S)}} X\text{-R} = \mathrm{S}$$

References

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Characterizing strong randomness via Martin-Löf randomness. *Annals of Pure and Applied Logic, vol. 163, no. 3, pp. 214-224 (2012)*

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Thank you very much!