Reuniting the antipodes: bringing together Nonstandard Analysis and Constructive Analysis

Sam Sanders[1]

CTFM, Feb. 18, 2013

[1]This research is generously supported by the John Templeton Foundation.

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3. (FUTURE work) An interpretation of Type Theory in NSA.

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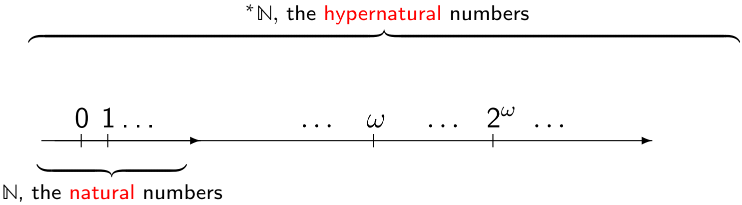

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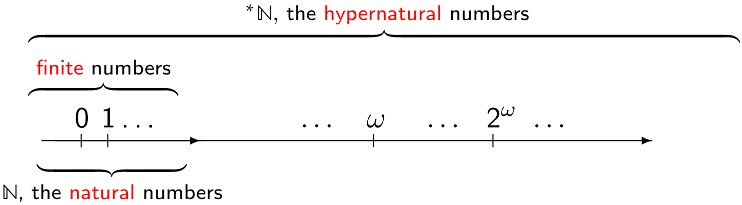

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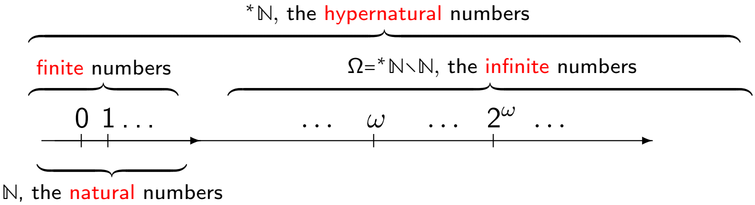

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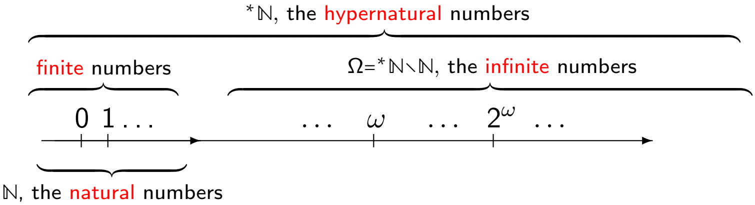

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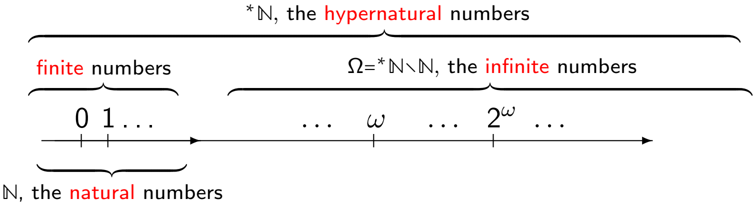

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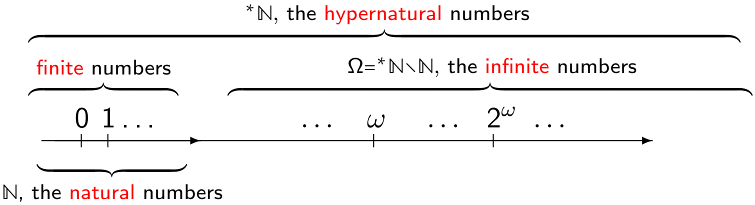

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${}^*I\Sigma_1 \vdash \Delta^0_1\text{-CA} \leftrightarrow \Omega\text{-CA}$ ‘(*Ω-invariance $\subseteq$ Turing comp.*)’

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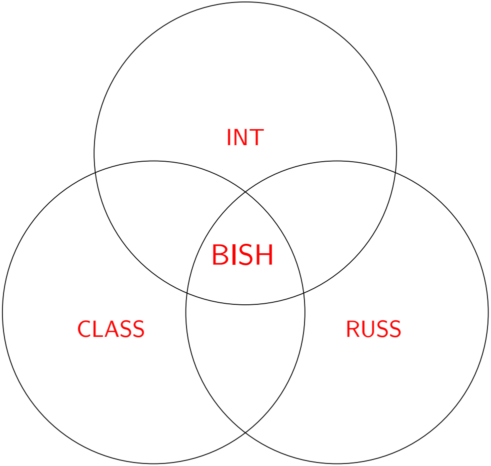

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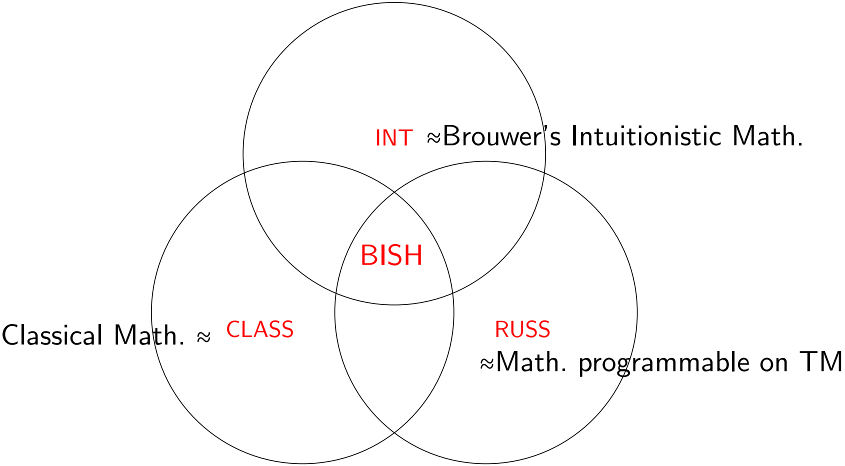

INT ≈Brouwer's Intuitionistic Math.
BISH
Classical Math. ≈ CLASS
RUSS
≈Math. programmable on TM

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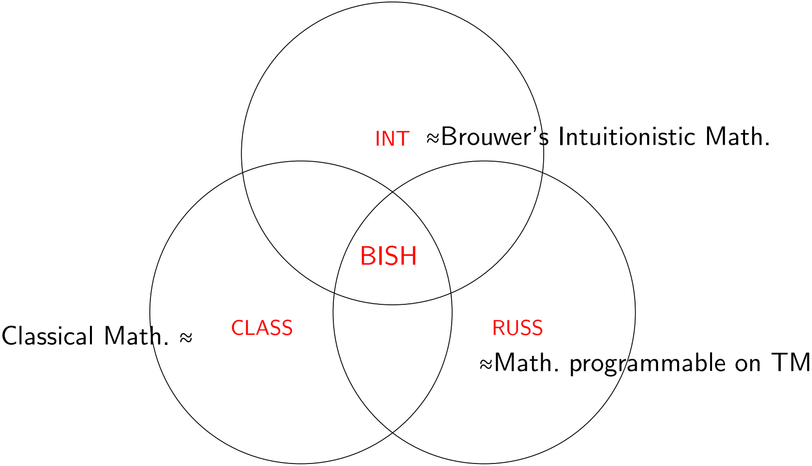

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6. $(\forall x \in A)P(x)$: for all x, $x \in A \to P(x)$.

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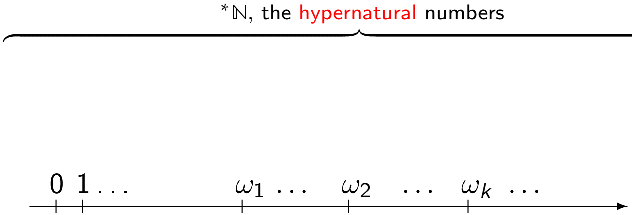

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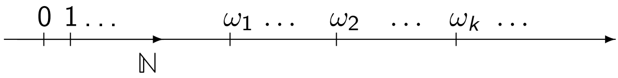

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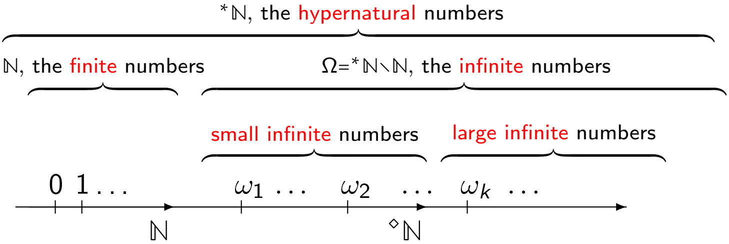

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BISH: Limited to formulas with proofs.
$\mathbb{NSA}$: Limited to formulas A such that $A \in \mathbb{T}$

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&\mathbb{B}(A \to B) := \mathbb{B}(A) \Rrightarrow \mathbb{B}(B)\\
&\mathbb{B}(\neg A) := \sim \mathbb{B}(A)
\end{aligned}
$$

$\mathbb{NSA}$ already has classical $\vee$, and $\mathbb{V}$ is a new connective called 'hyperdisjunction'.

$\mathbb{NSA}$ already has classical $\to$, and $\Rrightarrow$ is a new connective called 'hyperimplication'.

$\mathbb{NSA}$ already has classical $\neg$, and $\sim$ is a new connective called 'hypernegation'.

The interpretation $\mathbb{B}$ from BISH to classical $\mathbb{NSA}$

We define $\mathbb{B}$ which maps formulas of BISH to formulas of $\mathbb{NSA}$.

Let A, B be formulas in the language of BISH.

$$\begin{aligned}
&\mathbb{B}(A) := A \quad (A \in \Delta_0)\\
&\mathbb{B}(A \vee B) := \mathbb{B}(A) \mathbb{V} \mathbb{B}(B)\\
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&\mathbb{B}\big((\exists x)A(x)\big) := (\tilde{\exists} x)\mathbb{B}(A(x))
\end{aligned}$$

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&\mathbb{B}\big((\forall x)A(x)\big) := (\forall x)\mathbb{B}(A(x))
\end{aligned}
$$

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Lost in translation

Lost in translation

BISH (based on BHK)

Lost in translation

BISH (based on BHK)

$\mathbb{NSA}$ (based on CL)

Lost in translation

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
Central: algorithm and proof	

Lost in translation

BISH (based on BHK)

Central: algorithm and proof

$A \vee B$:
an algo yields a proof of A or of B

$\mathbb{NSA}$ (based on CL)

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$A \Rightarrow B$: $\big[A \wedge [A \in \mathbb{T}]\big] \to \big[B \wedge [B \in \mathbb{T}]\big]$

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'$A \in \mathbb{T}$' means 'A satisfies Transfer'.

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BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
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$A \vee B$: an algo yields a proof of A or of B	$A \mathbb{V} B$: There is Ω-invariant $\psi(\vec{x},\omega)$ s.t. $$\psi(\vec{x},\omega) \to [A(\vec{x}) \wedge [A(\vec{x}) \in \mathbb{T}]]$$ $$\wedge$$ $$\neg\psi(\vec{x},\omega) \to [B(\vec{x}) \wedge [B(\vec{x}) \in \mathbb{T}]]$$
$A \to B$: an algo converts a proof of A to a proof of B	$A \Rrightarrow B$: $\big[A \wedge [A \in \mathbb{T}]\big] \to \big[B \wedge [B \in \mathbb{T}]\big]$

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A	$A \in \mathbb{T}$
$(\forall n \in \mathbb{N})\varphi(n)$	$(\forall n \in \mathbb{N})\varphi(n) \to (\forall n \in {}^*\mathbb{N})\varphi(n)$

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$$\sim[(\forall n \in \mathbb{N})A(n)] \equiv (\exists n \in {}^{\circ}\mathbb{N}){\sim}A(n)$$

WEAKER than $(\exists n \in \mathbb{N}){\sim}A(n)$.

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$\neg[(\forall n \in \mathbb{N})A(n)]$ is WEAKER than $(\exists n \in \mathbb{N})\neg A(n)$.

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WHY is this a good/faithful/reasonable/... translation?

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BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
Central: algorithm and proof	Central: Ω-invariance and Transfer ($\mathbb{T}$)
$A \vee B$: an algo yields a proof of A or of B	$A \mathbb{V} B$: There is Ω-invariant $\psi(\vec{x},\omega)$ s.t. $\psi(\vec{x},\omega) \to [A(\vec{x}) \wedge [A(\vec{x}) \in \mathbb{T}]]$ $\wedge$ $\neg\psi(\vec{x},\omega) \to [B(\vec{x}) \wedge [B(\vec{x}) \in \mathbb{T}]]$
$A \to B$: an algo converts a proof of A to a proof of B	$A \Rightarrow B$: $\big[A \wedge [A \in \mathbb{T}]\big] \to \big[B \wedge [B \in \mathbb{T}]\big]$
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$(\exists x)A(x)$: an algo computes x_0 such that $A(x_0)$	$(\tilde{\exists} x)A(x)$: "an Ω-inv. proc. computes x_0 such that $A(x_0)$"

WHY is this a good/faithful/reasonable/... translation?

BECAUSE it preserves all essential features of BISH (e.g. CRM)

Constructive Reverse Mathematics

Constructive Reverse Mathematics

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	

Constructive Reverse Mathematics

BISH (based on BHK) | $\mathbb{NSA}$ (based on CL)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$

CIT: Cantor intersection thm

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$\Longleftrightarrow$

$\mathbb{NSA}$ (based on CL)

unavailable Transfer Principle

$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} {\sim}P$

$\Longleftrightarrow$

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non-constructive/non-algorithmic	unavailable Transfer Principle
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
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MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
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$\updownarrow$	$\Longleftrightarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x > 0))$
$\updownarrow$	$\Longleftrightarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo)	$\Longleftrightarrow$ (limit computed by Ω-inv. proc.)
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm
(point in intersection computed by algo)	
	(point in intersection computed by Ω-inv. proc.)

Constructive Reverse Mathematics

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LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

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$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} {\sim} P$

$\Longleftrightarrow$

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$\Longleftrightarrow$

$\mathbb{MCT}$: monotone convergence thm

$\Longleftrightarrow$ (limit computed by Ω-inv. proc.)

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$\Longleftrightarrow$

Π_1-TRANS^{SET}

$(\forall n \in \mathbb{N})\varphi(n, \vec{X}) \to (\forall n \in {}^*\mathbb{N})\varphi(n, {}^*\vec{X})$

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non-constructive/non-algorithmic	unavailable Transfer Principle
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	$\Longleftrightarrow$
	Π_1-TRANS^{SET}
	$(\forall n \in \mathbb{N})\varphi(n, \vec{X}) \rightarrow (\forall n \in {}^*\mathbb{N})\varphi(n, {}^*\vec{X})$
	$\Longleftrightarrow$
	$(\forall x \in \mathbb{R})(x \approx 0 \Rightarrow x = 0)$

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	$\Longleftrightarrow$
	Π_1-TRANS^{SET}
	$(\forall n \in \mathbb{N})\varphi(n, \vec{X}) \to (\forall n \in {}^*\mathbb{N})\varphi(n, {}^*\vec{X})$
	$\Longleftrightarrow$
	$(\forall x \in \mathbb{R})(x \approx 0 \Rightarrow x = 0)$

$\mathbb{NSA}$ does prove $(\forall \delta \in \mathbb{R})\big[\delta > 0 \Rightarrow (x > 0) \mathbb{V} (x < \delta)\big]$.

BISH does prove $(\forall \delta \in \mathbb{R})\big[\delta > 0 \to (x > 0) \vee (x < \delta)\big]$.

Constructive Reverse Mathematics II

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\updownarrow$

WKL

$\mathbb{NSA}$ (based on CL)

unavailable Transfer Principle

Constructive Reverse Mathematics II

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	unavailable Transfer Principle
LLPO	$\mathbb{LLPO}$
For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$	For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rightarrow \sim P \mathbb{V} \sim Q$
$\updownarrow$	$\Longleftrightarrow$
LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$	$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \mathbb{V} x \leq 0)$
$\updownarrow$	$\Longleftrightarrow$
NIL	$\mathbb{NIL}$
$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$	$(\forall x, y \in \mathbb{R})(xy = 0 \Rightarrow x = 0 \mathbb{V} y = 0)$
$\updownarrow$	$\Longleftrightarrow$
IVT: Intermediate value theorem	$\mathbb{IVT}$: Intermediate value theorem
$\updownarrow$	
WKL	$\Longleftrightarrow$ $\mathbb{WKL}$

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WKL	$\Longleftrightarrow$ $\mathbb{WKL}$ $\Longleftrightarrow$ $\vee$-Transfer

$$(\forall \Phi \in \Pi_1^0)(\forall \alpha, \beta \in 2^{\mathbb{N}})(\Phi(\alpha) \vee \Phi(\beta) \Rightarrow \Phi(\alpha) \mathbb{V} \Phi(\beta))$$

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

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$\Longleftrightarrow$

$\mathbb{IVT}$: Intermediate value theorem

$(\tilde{\exists} z \in [0,1])(f(z) = 0) \equiv$???

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BISH and $\mathbb{NSA}$ can prove $(\forall k \in \mathbb{N})(\exists x_0 \in [0,1])(|f(x_0)| < 1/k)$.

Constructive Reverse Mathematics III

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BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	Transfer Principle
MP: For $P \in \Sigma_1$, $\neg\neg P \to P$	$\mathbb{MP}$: For $P \in \Sigma_1$, $\sim\sim P \Rightarrow P$
$\updownarrow$	$\Longleftrightarrow$
MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \to x > 0)$	$\mathbb{MPR}$: $(\forall x \in \mathbb{R})(\sim\sim(x > 0) \Rightarrow x > 0)$
$\updownarrow$	$\Longleftrightarrow$
EXT: the extensionality theorem	$\mathbb{EXT}$: the extensionality theorem
WLPO: For $P \in \Sigma_1$, $\neg\neg P \vee \neg P$	$\mathbb{WLPO}$: For $P \in \Sigma_1$, $\sim\sim P \mathbb{V} \sim P$
$\updownarrow$	$\Longleftrightarrow$
WLPR: $(\forall x \in \mathbb{R})\big[\neg\neg(x > 0) \vee \neg(x > 0)\big]$	$\mathbb{WLPR}$: $(\forall x \in \mathbb{R})\big[\sim\sim(x > 0) \mathbb{V} \sim(x > 0)\big]$
$\updownarrow$	$\Longleftrightarrow$
DISC: A discontinuous $2^{\mathbb{N}} \to \mathbb{N}$-function exists.	$\mathbb{DISC}$: A discontinuous $2^{\mathbb{N}} \to \mathbb{N}$-function exists.

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A formula ψ is Δ_1 if $\psi \iff (\exists n \in \mathbb{N})\varphi_1(n) \iff (\forall m \in \mathbb{N})\varphi_2(m)$.

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$\mathbb{NSA}$ *does not prove* $\psi \vee \sim\psi$ *for all* $\psi \in \Delta_1$.

Constructive Reverse Mathematics IV

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BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
$\text{LPO} \leftrightarrow \text{MP}+\text{WLPO}$	$\mathbb{LPO} \Longleftrightarrow \mathbb{MP} + \mathbb{WLPO}$
$\text{MP} \leftrightarrow \text{WMP} + \text{MP}^\vee$	$\mathbb{MP} \Longleftrightarrow \mathbb{WMP} + \mathbb{MP}^\vee$
$\text{WLPO} \rightarrow \text{LLPO}$	$\mathbb{WLPO} \Rrightarrow \mathbb{LLPO}$
$\text{LLPO} \rightarrow \text{MP}^\vee$	$\mathbb{LLPO} \Rrightarrow \mathbb{MP}^\vee$
$\text{LPO} \rightarrow \text{BD-N}$	$\mathbb{LPO} \Rrightarrow \mathbb{BD}\text{-}\mathbb{N}$
$\text{LLPO} \rightarrow \text{FAN}_\Delta$	$\mathbb{LLPO} \Rrightarrow \mathbb{FAN}_\Delta$
$\text{LLPO} \leftrightarrow \text{WKL}$	$\mathbb{LLPO} \Longleftrightarrow \mathbb{WKL}$

Reuniting the antipodes: $\mathbb{NSA} \approx$ BISH

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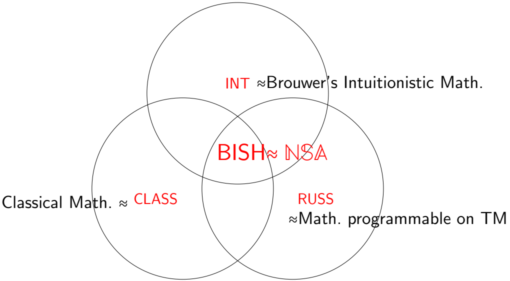

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ALGORITHM

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$=$ Built up from the ground out of basic elements.

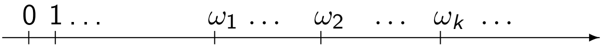
$=$ Independence of the choice of ideal object.

Why?

The nonstandard view: $\mathbb{N} \subset {}^{\diamond}\mathbb{N} \subset {}^{*}\mathbb{N}$

Why?

$\overbrace{\hspace{20em}}^{{}^{*}\mathbb{N},\ \text{the hypernatural numbers}}$



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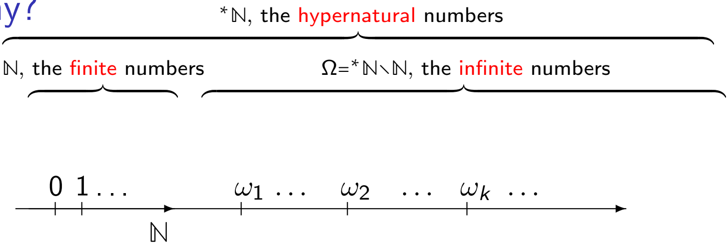

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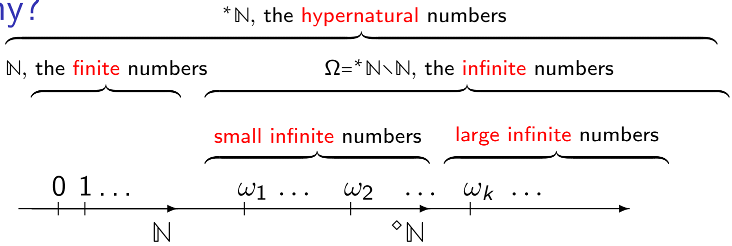

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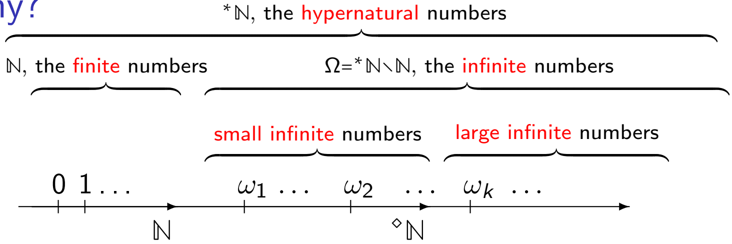

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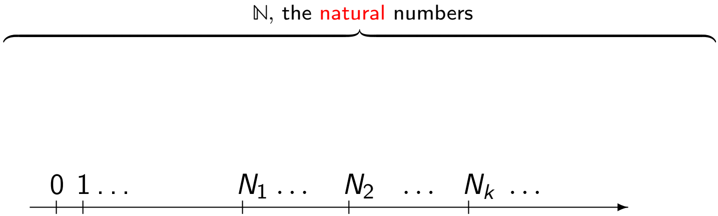

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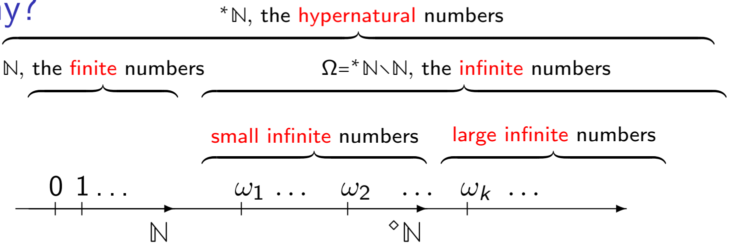

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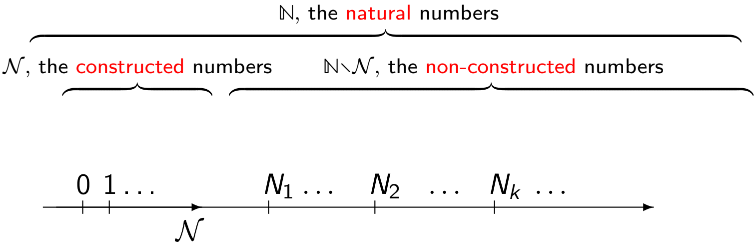

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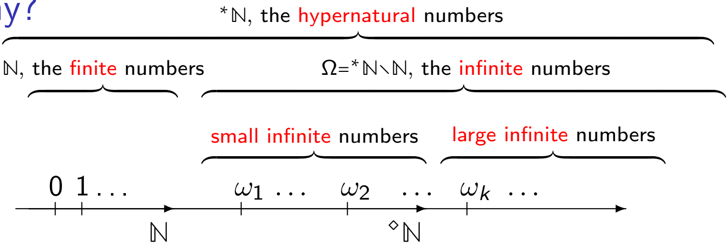

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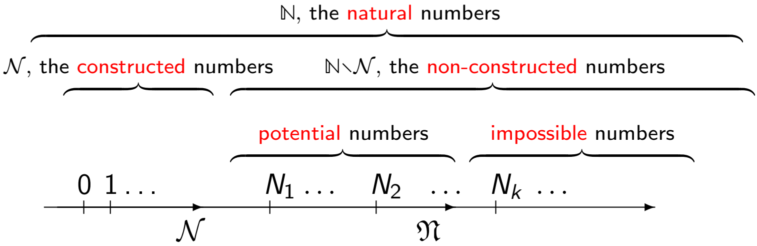

Future work: Type Theory

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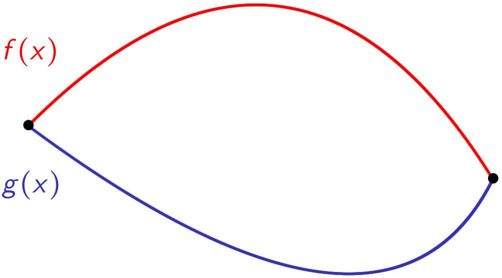

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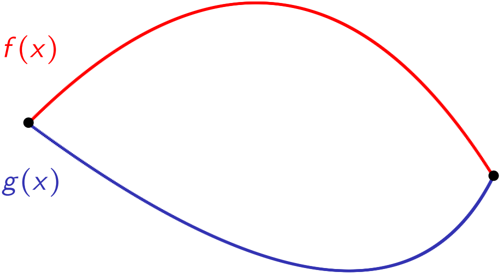

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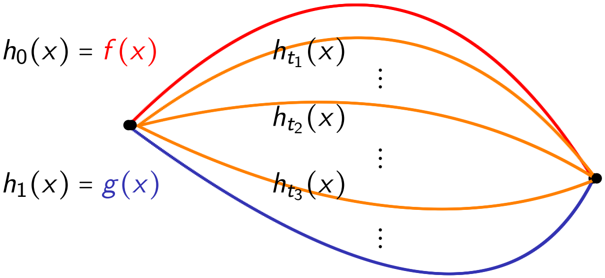

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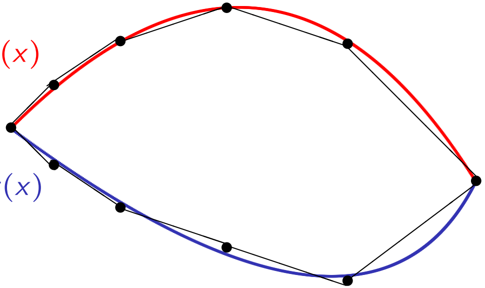

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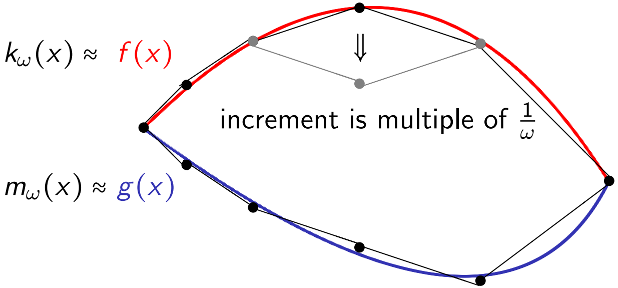

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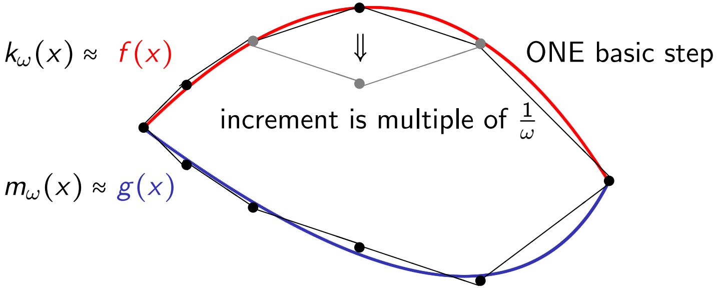

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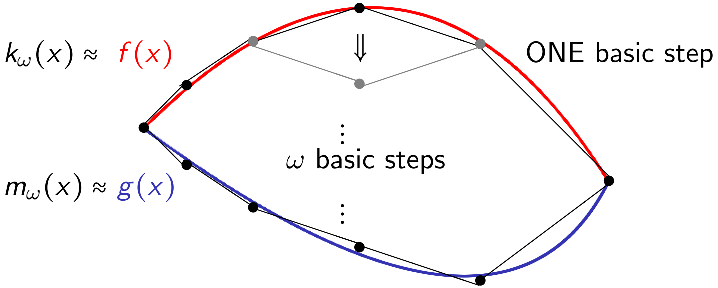

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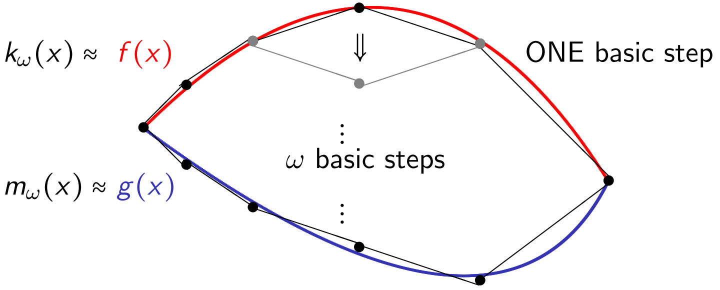

Independent of the choice of ω

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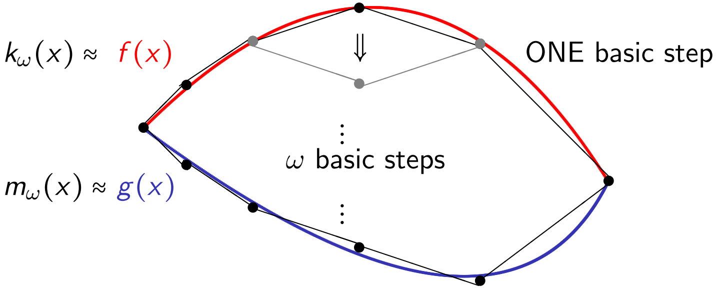
Homotopy: $\approx$ Ω-invariant broken-line transformation $h_{\omega,t}$ of f to g.



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A mathematical result with physical meaning will not depend on the choice of infinite number/infinitesimal used, i.e. it is Ω-invariant.

Final Thoughts

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...there are good reasons to believe that Nonstandard Analysis, in some version or other, will be the analysis of the future.

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Any questions?