Proofs, computations and analysis

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Motivation

Algorithms are viewed as one aspect of proofs in (constructive) analysis. A corresponding program (i.e., a term t in the underlying language) can be extracted from a proof of A, and a proof that t “realizes” A can be generated ($\Rightarrow$ automatic verification).

Data: From free algebras, given by their constructors. Examples:

- finite or infinite lists of signed digits -1, 0, 1 (i.e., reals as streams),
- possibly non well-founded alternating read-write trees (representing uniformly continuous functions).

Tools

- Decorations: $\to^{\mathrm{c}}, \forall^{\mathrm{c}}$ (short: $\to, \forall$) and $\to^{\mathrm{nc}}, \forall^{\mathrm{nc}}$ for removal of abstract data, and fine-tuning.
- Nested inductive/coinductive definitions of predicates. Their clauses give rise to free algebras. Only here computational content arises.

Computable functionals

- Types: $\iota \mid \rho \to \sigma$. Base types ι: free algebras (e.g., **N**), given by their signature.
- Functionals seen as limits of finite approximations: ideals (Kreisel, Scott, Ershov).
- Computable functionals are r.e. sets of finite approximations (example: fixed point functional).
- Functionals are partial. Total functionals are defined (by induction over the types).

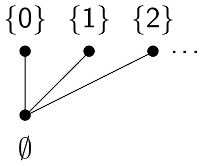
Information systems $\mathbf{C}_\rho$ for partial continuous functionals

- Types ρ, σ, τ: from algebras ι by $\rho \to \sigma$.
- $\mathbf{C}_\rho := (C_\rho, \mathrm{Con}_\rho, \vdash_\rho)$.
- Tokens $a \in C_\rho$ (= atomic pieces of information): constructor trees $\mathrm{C}a_1^*, \dots a_n^*$ with a_i^* a token or $*$. Example: $\mathrm{S}(\mathrm{S}*)$.
- Formal neighborhoods $U \in \mathrm{Con}_\rho$: $\{a_1, \dots, a_n\}$, consistent.
- Entailment $U \vdash_\rho a$.

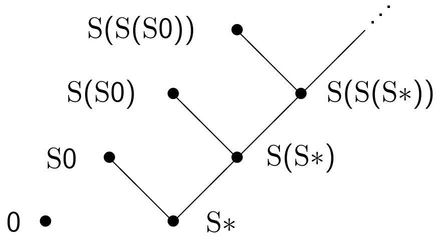
Ideals $x \in |\mathbf{C}_\rho|$ ("points", here: partial continuous functionals): consistent deductively closed sets of tokens.

Flat or non flat algebras?

- Flat:



- Non flat:



Non flat!

- Every constructor C generates an ideal in the function space: $r_{\mathrm{C}} := \{\, (U, \mathrm{C}a^*) \mid U \vdash a^* \,\}$. Associated continuous map:

$$|r_{\mathrm{C}}|(x) = \{\, \mathrm{C}a^* \mid \exists_{U \subseteq x}(U \vdash a^*) \,\}.$$

- Constructors are injective and have disjoint ranges:

$$|r_{\mathrm{C}}|(\vec{x}\,) \subseteq |r_{\mathrm{C}}|(\vec{y}\,) \leftrightarrow \vec{x} \subseteq \vec{y},$$
$$|r_{\mathrm{C}_1}|(\vec{x}\,) \cap |r_{\mathrm{C}_2}|(\vec{y}\,) = \emptyset.$$

- Both properties are false for flat information systems (for them, by monotonicity, constructors need to be strict).

$$|r_{\mathrm{C}}|(\emptyset, y) = \emptyset = |r_{\mathrm{C}}|(x, \emptyset),$$
$$|r_{\mathrm{C}_1}|(\emptyset) = \emptyset = |r_{\mathrm{C}_2}|(\emptyset).$$

A theory of computable functionals, TCF

- A variant of HA^ω.
- Variables range over arbitrary partial continuous functionals.
- Constants for (partial) computable functionals, defined by equations.
- Inductively and coinductively defined predicates. Totality for ground types inductively defined.
- Induction := elimination (or least-fixed-point) axiom for a totality predicate.
- Coinduction := greatest-fixed-point axiom for a coinductively defined predicate.

Relation to type theory

- Main difference: partial functionals are first class citizens.
- Minimal logic: $\to, \forall$ only. $=$ (Leibniz), $\exists$, $\vee$, $\wedge$ (Martin-Löf) inductively defined.
- $\bot := (\mathrm{False} = \mathrm{True})$. Ex-falso-quodlibet: $\bot \to A$ provable.
- Classical logic as a fragment: $\tilde{\exists}_x A$ defined by $\neg \forall_x \neg A$.

Realizability interpretation

- Define a formula $t \mathbf{r} A$, for A a formula and t a term in T^+.
- From a proof M we can extract its computational content, a term $\mathrm{et}(M)$.
- Soundness theorem:
  If M proves A, then $\mathrm{et}(M) \mathbf{r} A$ can be proved.
- Decorations: $\to^{\mathrm{c}}, \forall^{\mathrm{c}}$ (short: $\to, \forall$) and $\to^{\mathrm{nc}}, \forall^{\mathrm{nc}}$ for removal of abstract data, and fine-tuning:

$$\begin{aligned}
t \mathbf{r} (A \to^{\mathrm{c}} B) &:= \forall_x (x \mathbf{r} A \to tx \mathbf{r} B), \\
t \mathbf{r} (A \to^{\mathrm{nc}} B) &:= \forall_x (x \mathbf{r} A \to t \mathbf{r} B), \\
t \mathbf{r} (\forall^{\mathrm{c}}_x A) &:= \forall_x (tx \mathbf{r} A), \\
t \mathbf{r} (\forall^{\mathrm{nc}}_x A) &:= \forall_x (t \mathbf{r} A).
\end{aligned}$$

Example: decorating the existential quantifier

- $\exists_x A$ is inductively defined by the clause

$$\forall_x (A \to \exists_x A)$$

with least-fixed-point axiom

$$\exists_x A \to \forall_x (A \to P) \to P.$$

- Decoration leads to variants $\exists^{\mathrm{d}}, \exists^{\mathrm{l}}, \exists^{\mathrm{r}}, \exists^{\mathrm{u}}$ (d for "double", l for "left", r for "right" and u for "uniform").

$$\begin{array}{ll} \forall^{\mathrm{c}}_x (A \to^{\mathrm{c}} \exists^{\mathrm{d}}_x A), & \exists^{\mathrm{d}}_x A \to^{\mathrm{c}} \forall^{\mathrm{c}}_x (A \to^{\mathrm{c}} P) \to^{\mathrm{c}} P, \\ \forall^{\mathrm{nc}}_x (A \to^{\mathrm{c}} \exists^{\mathrm{r}}_x A), & \exists^{\mathrm{r}}_x A \to^{\mathrm{c}} \forall^{\mathrm{nc}}_x (A \to^{\mathrm{c}} P) \to^{\mathrm{c}} P. \end{array}$$

Practical aspects

- We need formalized proofs, to allow machine extraction.
- Can't take a proof assistant from the shelf: none fits TCF.

Minlog (`http://www.minlog-system.de`)

- Natural deduction for $\to, \forall$, plus inductively and coinductively defined predicates.
- Partial functionals are first class citizens.
- Allows type and predicate parameters (for abstract developments: groups, fields, reals, . . .).

Uniformly continuous functions

Based on work of Ulrich Berger (2009).

- Extraction from a proof dealing with abstract uniformly continuous functions.
- Data representing uniformly continuous functions: base type cototal ideals.
- The extracted term will involve corecursion.

Type-1 representation of uniformly continuous functions

For contrast: a type-1 represented function $f\colon [-1,1] \to [-1,1]$ is given by

- an approximating map $h\colon [-1,1] \cap \mathbb{Q} \to \mathbb{N} \to \mathbb{Q}$,
- bounds $N, M \in \mathbb{N}$ with $\forall_{a\in[-1,1]}\forall_n (N \leq h(a,n) \leq M)$, and
- a weakly increasing map $\alpha\colon \mathbb{N} \to \mathbb{N}$ such that $(h(a,n))_n$ is a Cauchy sequence with (uniform) modulus α, i.e.,

$$\forall_{a\in[-1,1]}\forall_k\forall_{n,m\geq\alpha(k)}(|h(a,n) - h(a,m)| \leq 2^{-k}).$$

f is (uniformly) continuous if we have a weakly increasing modulus $\omega\colon \mathbb{N} \to \mathbb{N}$ such that

$$\forall_k\forall_{a,b\in[-1,1]}\forall_{n\geq\alpha(k)}(|a-b| \leq 2^{-\omega(k)+1} \to |h(a,n)-h(b,n)| \leq 2^{-k}).$$

Application $f(x)$

Application of f given by h, α and modulus ω to $x := ((a_n)_n, M)$:

$$f(x) := (h(a_n, n))_n$$

with Cauchy modulus $\max(\alpha(k+2), M(\omega(k+1)-1))$.

Intermediate value theorem

Let $a < b$ be rationals. If $f\colon [a, b] \to \mathbb{R}$ is continuous with $f(a) \le 0 \le f(b)$, and with a uniform lower bound on its slope, then we can find $x \in [a, b]$ such that $f(x) = 0$.

Proof sketch.

1. Approximate Splitting Principle. Let x, y, z be given with $x < y$. Then $z \le y$ or $x \le z$.
2. IVTAux. Assume $a \le c < d \le b$, say $2^{-n} < d - c$, and $f(c) \le 0 \le f(d)$. Construct c_1, d_1 with $d_1 - c_1 = \frac{2}{3}(d - c)$, such that $a \le c \le c_1 < d_1 \le d \le b$ and $f(c_1) \le 0 \le f(d_1)$.
3. IVTcds. Iterate the step $c, d \mapsto c_1, d_1$ in IVTAux.

Let $x = (c_n)_n$ and $y = (d_n)_n$ with the obvious modulus. As f is continuous, $f(x) = 0 = f(y)$ for the real number $x = y$. $\square$

Extracted term

```
[k0]
 left((cDC rat@@rat)(1@2)
      ([n1]
        (cId rat@@rat=>rat@@rat)
        ([cd3]
          [let cd4
            ((2#3)*left cd3+(1#3)*right cd3@
             (1#3)*left cd3+(2#3)*right cd3)
            [if (0<=(left cd4*left cd4-2+
                     (right cd4*right cd4-2))/2)
             (left cd3@right cd4)
             (left cd4@right cd3)]]))
      (IntToNat(2*k0)))
```

where cDC is a from of the recursion operator.

Free algebra $\mathbf{J}$ of intervals

- $\mathbf{SD} := \{-1, 0, 1\}$ signed digits (or $\{L, M, R\}$).
- $\mathbf{J}$ free algebra of intervals. Constructors

$\mathbb{I}$	the interval $[-1, 1]$,
$\mathrm{C} \colon \mathbf{SD} \to \mathbf{J} \to \mathbf{J}$	left, middle, right half.

  Write $\mathrm{C}_d x$ for $\mathrm{C} d x$.
- $\mathrm{C}_1 \mathbb{I}$ denotes $[0, 1]$.
- $\mathrm{C}_0 \mathbb{I}$ denotes $[-\frac{1}{2}, \frac{1}{2}]$.
- $\mathrm{C}_0(\mathrm{C}_{-1} \mathbb{I})$ denotes $[-\frac{1}{2}, 0]$.

$\mathrm{C}_{d_0}(\mathrm{C}_{d_1} \dots (\mathrm{C}_{d_{k-1}} \mathbb{I}) \dots)$ denotes the interval in $[-1, 1]$ whose reals have a signed digit representation starting with $d_0 d_1 \dots d_{k-1}$.

- We consider ideals $x \in |\mathbf{C}_{\mathbf{J}}|$.

Total and cototal ideals of base type

Generally:

- Cototal ideals x: every token (i.e., constructor tree) $P(*) \in x$ has a "$\succ_1$-successor" $P(\mathrm{C}\vec{*}) \in x$.
- Total ideals: the cototal ones with $\succ_1$ well-founded.

Examples:

- Total ideals of **J**:

$$\mathbb{I}_{\frac{i}{2^k},k} := [\frac{i}{2^k} - \frac{1}{2^k}, \frac{i}{2^k} + \frac{1}{2^k}] \qquad \text{for } -2^k < i < 2^k.$$

- Cototal ideals of **J**: reals in $[-1, 1]$, in (non-unique) stream representation using signed digits $-1, 0, 1$.

Corecursion

- The conversion rules for $\mathcal{R}$ with total ideals as recursion arguments work from the leaves towards the root, and terminate because total ideals are well-founded.
- For cototal ideals (streams) a similar operator is available to define functions with cototal ideals as values: corecursion.
- ${}^{\mathrm{co}}\mathcal{R}^{\tau}_{\mathbf{J}} \colon \tau \to (\tau \to \mathbf{U} + \mathbf{SD} \times (\mathbf{J} + \tau)) \to \mathbf{J}$ (**U** unit type).
- Conversion rule

$$\begin{aligned}
{}^{\mathrm{co}}\mathcal{R}^{\tau}_{\mathbf{J}} NM \mapsto [\textbf{case } (MN)^{\mathbf{U}+\mathbf{SD}\times(\mathbf{J}+\tau)} \textbf{ of} \\
&\mathrm{inl}\ \_ \mapsto \mathbb{I} \mid \\
&\mathrm{inr}\langle d, z\rangle \mapsto \mathrm{C}_d[\textbf{case } z^{\mathbf{J}+\tau} \textbf{ of} \\
&\qquad\qquad\qquad \mathrm{inl}\ \_ \mapsto \mathbb{I} \mid \\
&\qquad\qquad\qquad \mathrm{inr}\ u^{\tau} \mapsto {}^{\mathrm{co}}\mathcal{R}^{\tau}_{\mathbf{J}} uM]].
\end{aligned}$$

W and continuous real functions

- Consider a well-founded "read tree", i.e., a constructor tree built from R (ternary) with R_d at its leaves.
- The digit d at a leaf means that, after reading all input digits on the path leading to the leaf, the output d is written.
- Let $R_{d_1}, \dots, R_{d_n}$ be all leaves. At a leaf R_{d_i} continue with W (i.e., write d_i), and continue reading.
- Result: a "nested **R**(**W**)-total **W**-cototal" ideal, representing a uniformly continuous real function $f \colon \mathbb{I} \to \mathbb{I}$.

A read-write instruction

```
W      W      W
-      -      -
d      d      d
 \     |     /
   \   |   /      W          W
     \ | /        -          -
       o          d          d
        \         |         /
          \       |       /
            \     |     /
              \   |   /
                \ | /
                  o
```

$\mathbf{R}(\alpha) := \mu_\xi(\alpha \to \xi, \alpha \to \xi, \alpha \to \xi, \xi \to \xi \to \xi \to \xi)$ labelled read-and-finally-write-one-digit trees. Constructors:

$R_d \colon \alpha \to \mathbf{R}(\alpha)$	$(d \in \{-1, 0, 1\})$	finally write d & continue,
$R \colon \mathbf{R}(\alpha) \to \mathbf{R}(\alpha) \to \mathbf{R}(\alpha) \to \mathbf{R}(\alpha)$		read.

Using $\mathbf{R}(\alpha)$ define nested alternating read-write trees

$$\mathbf{W} := \mu_\xi(\xi, \mathbf{R}(\xi) \to \xi)$$

with constructors

$W_0 \colon \mathbf{W}$	Stop,
$W \colon \mathbf{R}(\mathbf{W}) \to \mathbf{W}$	Branch by applying a read-write instruction, and continue.

Want finite read-write instructions, but infinitely many alternations, via a “nested inductive/coinductive” definition.

$\mathrm{Read}(X)$

We give an inductive definition of a unary predicate $\mathrm{Read}(X)$ on functions f; it depends on a parameter X:

$$f[\mathbb{I}] \subseteq \mathbb{I}_d \to X(\mathrm{out}_d \circ f) \to \mathrm{Read}(X)f \quad (d \in \{-1, 0, 1\}),$$
$$(\mathrm{Read}(X)(f \circ \mathrm{in}_d))_{d\in\{-1,0,1\}} \to \mathrm{Read}(X)f.$$

with $\mathrm{in}_d(a) := \frac{a+d}{2}$ and $\mathrm{out}_d(a) := 2a - d$. The corresponding least-fixed-point axiom is

$\mathrm{Read}(X)f \to$
$(\forall^{\mathrm{nc}}_f(f[\mathbb{I}] \subseteq \mathbb{I}_d \to X(\mathrm{out}_d \circ f) \to Pf))_{d\in\{-1,0,1\}} \to$
$\forall^{\mathrm{nc}}_f((\mathrm{Read}(X)(f \circ \mathrm{in}_d))_{d\in\{-1,0,1\}} \to (P(f \circ \mathrm{in}_d))_{d\in\{-1,0,1\}} \to Pf) \to$
$Pf)$.

Write and its dual ${}^{\mathrm{co}}\mathrm{Write}$

Using $\mathrm{Read}(X)$ we give a nested inductive definition of another unary predicate Write by

$$\begin{aligned}&\mathrm{Write}(\mathrm{id}),\\ &\mathrm{Read}(\mathrm{Write})f \to \mathrm{Write}\, f.\end{aligned}$$

Its dual ${}^{\mathrm{co}}\mathrm{Write}$ is defined by

$${}^{\mathrm{co}}\mathrm{Write}\, f \to \mathrm{Eq}(f, \mathrm{id}) \vee \mathrm{Read}({}^{\mathrm{co}}\mathrm{Write})f.$$

The greatest-fixed-point axiom ${}^{\mathrm{co}}\mathrm{Write}^{+}$ is

$$Pf \to \forall_f^{\mathrm{nc}}(Pf \to \mathrm{Eq}(f, \mathrm{id}) \vee \mathrm{Read}({}^{\mathrm{co}}\mathrm{Write} \vee P)f) \to {}^{\mathrm{co}}\mathrm{Write}\, f.$$

${}^{\mathrm{co}}\mathrm{Write}$ is an example of a nested inductive/coinductive predicate.

Define

$$B_{l,k}f := \forall_{p \in \mathbb{I}} \exists_q (f[\mathbb{I}_{p,l}] \subseteq \mathbb{I}_{q,k}).$$
$$Cf := \forall_k \exists_l B_{l,k} f.$$

Theorem

$\forall^{\mathrm{nc}}_f (Cf \leftrightarrow {}^{\mathrm{co}}\mathrm{Write}\, f).$

Proof sketch for $\to$.

We use the greatest-fixed-point axiom ${}^{\mathrm{co}}\mathrm{Write}^+$ with $P := C$. Fix f; it suffices to show $Cf \to \mathrm{Read}({}^{\mathrm{co}}\mathrm{Write} \vee C)f$. Assume Cf. By definition we have an l such that $B_{l,2}f$. Prove

$$\forall_l \forall^{\mathrm{nc}}_f (B_{l,2}f \to Cf \to \mathrm{Read}({}^{\mathrm{co}}\mathrm{Write} \vee C)f)$$

by induction on l. □

Why is this useful?

Recall the Theorem: $\forall^{\mathrm{nc}}_f (Cf \leftrightarrow {}^{\mathrm{co}}\mathrm{Write}\, f)$.

A witness of ${}^{\mathrm{co}}\mathrm{Write}\, f$ is a nested alternating read-write tree. The theorem allows to switch to such (base type) data when proving properties of continuous functions.

Example: the composition $g \circ f$ of two continuous functions $f, g \colon \mathbb{I} \to \mathbb{I}$ is continuous.

The extracted term involves a corecursion operator with nested recursion operators.

Conclusion

TCF (theory of computable functionals) as a possible foundation for (constructive) exact real arithmetic.

- Simply typed theory, with “lazy” free algebras as base types ($\Rightarrow$ constructors are injective and have disjoint ranges).
- Variables range over partial continuous functionals.
- Constants denote computable functionals (:= r.e. ideals).
- Minimal logic ($\to$, $\forall$), plus inductive & coinductive definitions.
- Computational content in abstract theories.
- Decorations ($\to^{\mathrm{c}}, \forall^{\mathrm{c}}$ and $\to^{\mathrm{nc}}, \forall^{\mathrm{nc}}$) for removal of abstract data, and fine-tuning.
- A nested inductive/coinductive definition of alternating read-write trees representing (uniformly) continuous functions.
- Base type representation of continuous functions when extracting computational content from proofs.

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