Recursively defined trees and their maximal order types

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CTFM 2013

[2]Work related with a program between Michael Rathjen and Andreas Weiermann

Structure presentation

The theorem of Kruskal

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$\mathbb{T}$ *is a wpo.*

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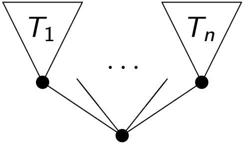
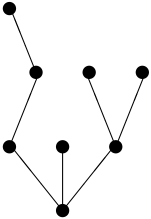
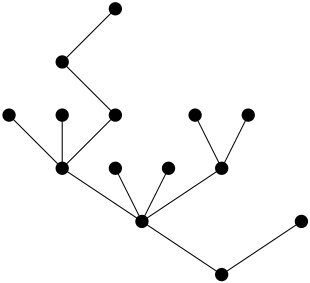


is also an element of $\mathbb{T}$.

Tree-embeddability



$\leq_{\mathbb{T}}$



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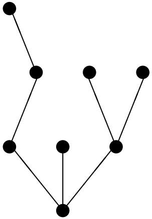

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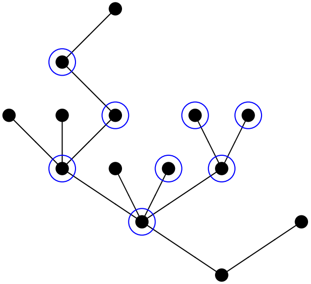

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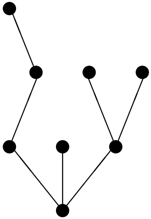

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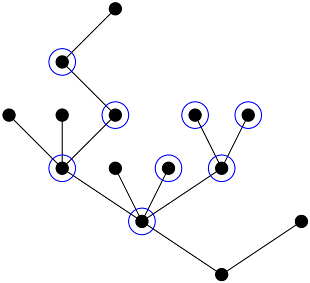

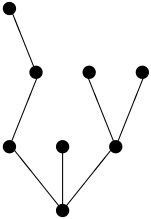

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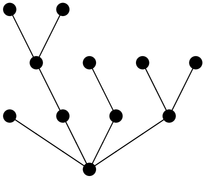

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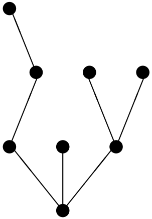

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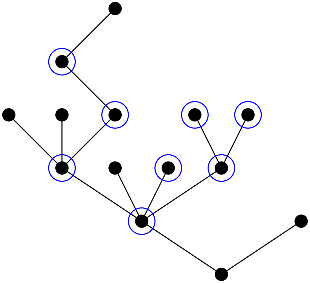

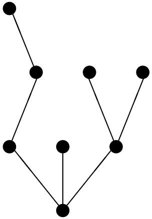

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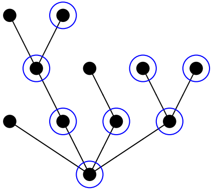
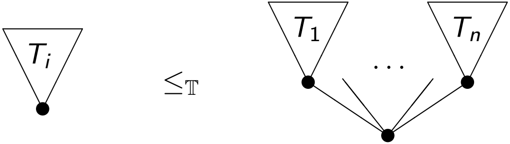


Tree-embeddability: definition



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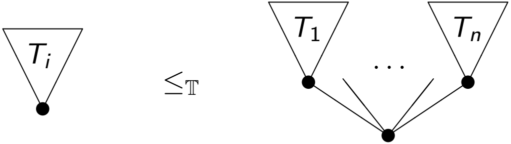
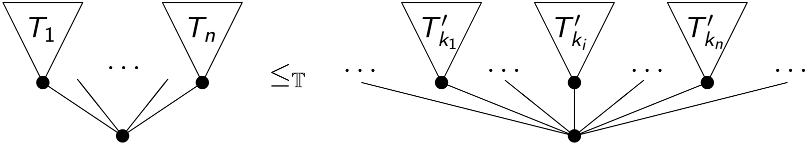


If $k_1 < k_2 < \cdots < k_n$ and $T_i \leq_{\mathbb{T}} T'_{k_i}$ for every i, then



The theorem of Kruskal

Theorem (Kruskal)

$\mathbb{T}$ *is a wpo.*

What is a wpo?

A well-partial-ordering (wpo) is a partial ordering that is

- well-founded,
- has no infinite antichain.

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Definition

A **well-partial-ordering** $(X, \leq_X)$ is a partial ordering such that for every infinite sequence $x_1, x_2, \ldots$ of elements in X, indices $i < j$ exists such that $x_i \leq_X x_j$.

The theorem of Kruskal

Theorem (Kruskal)

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$$=$$

Theorem (Kruskal)

For every infinite sequence $T_1, T_2, \ldots$ *of elements in* $\mathbb{T}$*, there exists indices* $i < j$ *such that* $T_i \leq_{\mathbb{T}} T_j$*.*

Theorem

$\mathbb{T}$ *is wpo.*

$$\Downarrow$$

New tree-class $\mathcal{T}(W)$

Theorem

$\mathcal{T}(W)$ *is wpo.*

Interested in:

- Is this theorem true?
- What is the maximal order type of $\mathcal{T}(W)$?
- Which theories T can (and which cannot) prove '$\mathcal{T}(W)$ is wpo'?

Why interested in this?

- Trying to obtain the strength of trees with gap-condition.
- A natural generalization of the notion 'tree' and of Kruskal's theorem.
- Relations with ordinal notation systems.

Maximal order type

Definition

The **maximal order type** of a well-partial-ordering $(X, \leq_X)$ is defined as

$$o(X, \leq_X) = \sup\{\alpha \mid \leq_X \subseteq \leq^+ \text{ with } \leq^+ \text{ a linear ordering on } X \text{ and } \alpha = otype(X, \leq^+)\}.$$

Every extension of a well-partial-ordering to a linear ordering is a well-ordering.

Let us introduce $\mathcal{T}(W)$!

Definition X^*: the Higman ordering

X^* is the set of finite sequence over X with the Higman ordering:

$$(x_1, \ldots, x_n) \leq^* (y_1, \ldots, y_m)$$
$$\Leftrightarrow \exists 1 \leq i_1 < \cdots < i_n \leq m \text{ such that } x_j \leq_X y_{i_j} \text{ for every } j = 1, \ldots, n.$$

Theorem

If X is a well-partial-ordering, then X^* is also a well-partial-ordering.

Definition X^*: the Higman ordering

Theorem (De Jongh & Parikh; D. Schmidt)

If X is a well-partial-ordering, then

$$o(X^*) = \begin{cases} \omega^{\omega^{o(X)+1}} & \text{if } o(X) \text{ is equal to } e+n \\ & \quad \text{with } e \text{ an epsilon number and } n<\omega, \\ \omega^{\omega^{o(X)-1}} & \text{if } o(X) \text{ is finite}, \\ \omega^{\omega^{o(X)}} & \text{otherwise.} \end{cases}$$

$W(X) = X^+$

$$W(X) = X^+ = X^* \backslash \{()\}.$$

Example: $\mathcal{T}(X^+)$

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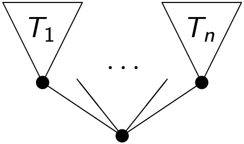

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Example: $\mathcal{T}(X^+)$

$\mathcal{T}(X^+)$ is the set of finite planar rooted trees:

- $\bullet$ is an element of $\mathcal{T}(X^+)$,
- If $(T_1, \ldots, T_n) \in (\mathcal{T}(X^+))^+$, then

$$\bullet[(T_1, \ldots, T_n)]$$

is also an element of $\mathcal{T}(X^+)$.

Recursively defined trees

Definition

$\mathcal{T}(W)$ is defined recursively as follows:

1. $\circ$ is an element of $\mathcal{T}(W)$.
2. If $w(T_1, \ldots, T_n)$ is an element of $W(\mathcal{T}(W))$, then $\circ[w(T_1, \ldots, T_n)]$ is an element of $\mathcal{T}(W)$.

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$W(X)$ satisfies:

- If X is a countable well-partial-ordering, then $W(X)$ is also a countable well-partial-ordering, hence

$$\forall X(WPO(X) \rightarrow WPO(W(X))).$$

- Elements of $W(X)$ are formal terms with entries in X.
- Equality $o(W(X)) = o(W(o(X)))$ can be proved by using an effective reification.

Recursively defined trees

Definition

We define $\leq_{\mathcal{T}(W)}$ on $\mathcal{T}(W)$ as follows:

1. $\circ \leq_{\mathcal{T}(W)} t$ for every t in $\mathcal{T}(W)$,
2. for every j: if $s \leq_{\mathcal{T}(W)} t_j$, then $s \leq_{\mathcal{T}(W)} \circ[w(t_1, \ldots, t_n)]$,
3. if $w(t_1, \ldots, t_n) \leq_{W(\mathcal{T}(W))} w'(t'_1, \ldots, t'_m)$, then $\circ[w(t_1, \ldots, t_n)] \leq_{\mathcal{T}(W)} \circ[w'(t'_1, \ldots, t'_m)]$.

Examples

If $W(X) = X$, then

$$\mathcal{T}(W) \cong \mathbb{N}.$$

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If $W(X) = X^+ = X^* \backslash \{()\}$, then
$$\mathcal{T}(W) \cong \mathbb{T}.$$

If $W(X) = (X \times X)^+$, then

$\mathcal{T}(W)$: gluing together of immediate subtrees in pairs

The maximal order type of $\mathcal{T}(W)$

Conjecture (Weiermann)

$\mathcal{T}(W)$ is a well-partial-ordering and

$$o(\mathcal{T}(W)) = \theta(o(W(\Omega))),$$

if $o(W(\Omega)) \geq \Omega^3$ and $o(W(\Omega)) \in dom(\theta)$.

The maximal order type of $\mathcal{T}(W)$

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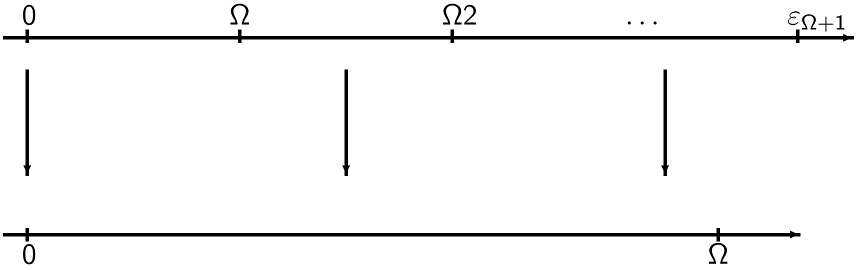
if $o(W(\Omega)) \geq \Omega^3$ and $o(W(\Omega)) \in dom(\theta)$.

If $W(X) = X^+$, then $\mathcal{T}(W) \cong \mathbb{T}$. Also, $o(W(\Omega)) = \omega^{\omega^{\Omega+1}} = \Omega^\omega$, hence

$$o(\mathbb{T}) = o(\mathcal{T}(W)) = \theta(o(W(\Omega))) = \theta(\Omega^\omega).$$

$\rightarrow$ generalization result Diana Schmidt!

The collapsing function $\theta : \varepsilon_{\Omega+1} \to \Omega$



Results about the maximal order type

$W(X)$	$o(\mathcal{T}(W))$
$M^{\diamond}(X \times X)$	$\theta(\Omega^{\Omega})$
$M(X \times X)$	$\theta(\Omega^{\Omega})$
$(X \times X)^*$	$\theta(\Omega^{\Omega^{\Omega}})$
$(X^*)^*$	$\theta(\Omega^{\Omega^{\Omega^{\omega}}})$
$B(X)$	$\theta(\varepsilon_{\Omega+1})$

Proof-theoretical strength

Theorem

$ACA_0 + (\Pi^1_1\text{–}CA_0)^- \nvdash$ *'$\mathcal{T}(B)$ is a wpo',*

$ACA_0 + (\Pi^1_1\text{–}CA_0)^- \vdash$ *'$\mathcal{T}(X^{\overbrace{*\cdots*}^{n}})$ is a wpo' ($n \in \mathbb{N}$).*

Theorem

$RCA_0 + (\Pi^1_1\text{–}CA_0)^- \vdash$ *'$\mathcal{T}(X^n)$ is wpo', for every $n \geq 2$.*

Ongoing:

$$RCA_0 + (\Pi^1_1\text{–}CA_0)^- \nvdash \text{'}\mathcal{T}(X^*) \text{ is wpo'}.$$

Interesting conjecture

Conjecture (Rathjen, Weiermann)

$|RCA_0^* + (\Pi_1^1 - CA_0)^-| = \varphi\omega 0.$

Conclusions

- Definition well-partial-ordering and Kruskal's theorem
- Definition maximal order type of a wpo
- Recursively defined trees
- Conjecture $o(\mathcal{T}(W)) = \theta(o(W(\Omega)))$
- Results on ordinal and proof-theoretical strength

Thank you for your attention!

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