Where closure under Turing jumps can replace elementarity between structures

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20 February, 2013

*My current appointment is funded by the John Templeton Foundation.

Nonstandard arithmetic

- The language $\mathscr{L}_{\mathsf{A}}$ is $\{0, 1, +, \times, <\}$.
- *Peano arithmetic* (PA) consists of the axioms for *discretely ordered semirings*, and the *induction axiom*

$$\theta(0) \wedge \forall x \big(\theta(x) \to \theta(x+1)\big) \to \forall x \; \theta(x)$$

  for each formula $\theta(x)$.
- A *nonstandard* model of PA is a model not isomorphic to ω.
- Skolem (1934) showed that nonstandard models exist.

The standard cut

Fix a nonstandard model $M \models \mathrm{PA}$.

- $\mathscr{L}_{\mathsf{A}}$ has terms for 0, 1, $1+1$, ...
- So M contains a copy of ω that is often called the *standard cut*.

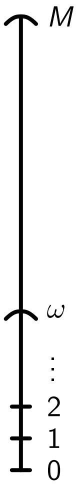
Kaye, Kossak, W. Adding standardness to nonstandard arithmetic. Forthcoming.

- Study the expanded structure (M, ω) in the language $\mathscr{L}_{\boldsymbol{\omega}} = \mathscr{L}_{\mathsf{A}} \cup \{\boldsymbol{\omega}\}$.

unary predicate

Why add $\boldsymbol{\omega}$?

- Nonstandard analysis
- Model theory
- Reverse mathematics



This talk

Where closure under Turing jumps can replace elementarity between structures

Plan

- Introduction
- Kanovei's Theorem (elementarity between structures)
- Variation (closure under Turing jumps)
- Conclusion

Motivation

$\mathrm{Th}\,\mathfrak{M}$ denotes the set of all sentences true in $\mathfrak{M}$.

Fix a nonstandard model $M \models \mathrm{PA}$.

- Gödel (1931) says $\mathrm{Th}(M) \not\leqslant_{\mathrm{T}} 0$.
- $\mathrm{Th}(M)$ can be "close to being recursive".
- $\mathrm{Th}(M)$ *represents* some nonrecursive set.
- $\mathrm{Th}(M, \omega) \geqslant_{\mathrm{T}} 0^{(n)}$ for all $n \in \omega$.
- $0^{(n)}$ is parameter-free definable in (M, ω) for all $n \in \omega$.

Question

Can $0^{(\omega)}$ be parameter-free definable in (M, ω)?

Answer (Kanovei 1996)

Yes, when M is an elementary extension of ω.

Elementary extensions

Definition

An extension $M \supseteq N$ is *elementary* if

$$M \models \varphi(\bar{n}) \quad \Leftrightarrow \quad N \models \varphi(\bar{n}).$$

for all formulas φ and all $\bar{n} \in N$. We write $M \succ N$ for this.

Theorem (Kanovei 1996)

If $M \succ \omega$, then $0^{(\omega)}$ is parameter-free definable in (M, ω).

Proof outline

- Recall $0^{(\omega)} \equiv_{\mathrm{T}} \mathrm{Th}(\omega)$.
- Our formula $\tau(\sigma)$ defining $\mathrm{Th}(\omega)$ in (M, ω) says

‘there is a certificate for the truth of σ in ω’.

What certifies truth?

Example

Let $\sigma = \forall x \; \exists y \; \varphi(x, y)$, where φ is quantifier-free.

$$\dfrac{\dfrac{\varphi(0, n_0)}{\exists y \; \varphi(0, y)} \qquad \dfrac{\varphi(1, n_1)}{\exists y \; \varphi(1, y)} \qquad \dfrac{\varphi(2, n_2)}{\exists y \; \varphi(2, y)} \qquad \begin{matrix} \cdots \\ \cdots \end{matrix}}{\forall x \; \exists y \; \varphi(x, y)}$$

Definition

A set of sentences C is a *truth certificate* if the following hold.

(a) If a quantifier-free $\varphi \in C$, then φ is true in ω.

(b) If $\forall x \; \varphi(x) \in C$, then $\varphi(m) \in C$ for all $m \in \omega$.

(c) If $\exists y \; \psi(y) \in C$, then $\psi(n) \in C$ for some $n \in \omega$.

Recall $\tau(\sigma)$ is meant to define $\mathrm{Th}(\omega)$ in M

definable/coded

- $\tau(\sigma)$ says 'there is a truth certificate C containing σ'.
- For a sentence σ, if $(M, \omega) \models \tau(\sigma)$, then $\omega \models \sigma$.

Proposition

If $\omega \models \sigma$ and $M \succ \omega$, then $(M, \omega) \models \tau(\sigma)$.

Proof sketch

Consider $\sigma = \forall x\ \exists y\ \varphi(x, y)$, where φ is quantifier-free. Define

$$\begin{aligned} P_0 &= \left\{(m,n) \in \omega^2 : \omega \models \varphi(m,n)\right\} \\ &= \left\{(m,n) \in \omega^2 : M \models \varphi(m,n)\right\} && \text{by elementarity,} \\ P_1 &= \left\{m \in \omega : \omega \models \exists y\ \varphi(m,y)\right\} \\ &= \left\{m \in \omega : M \models \exists y\ \varphi(m,y)\right\} && \text{by elementarity.} \end{aligned}$$

Then $C = \{\sigma\} \cup \{\exists y\ \varphi(m,y) : m \in P_1\} \cup \{\varphi(m,n) : (m,n) \in P_0\}$ is a truth certificate containing σ in M, because $\omega \models \sigma$. □

Arithmetical comprehension

$M \models \mathrm{I}\Delta_0 + \exp$

Recall
$$\begin{aligned} P_0 &= \big\{(m,n) \in \omega^2 : \omega \models \varphi(m,n)\big\} \\ &= \big\{(m,n) \in \omega^2 : M \models \varphi(m,n)\big\}, \\ P_1 &= \big\{m \in \omega : \omega \models \exists y \ \varphi(m,y)\big\} \\ &= \big\{m \in \omega : \exists y {\in} \omega \ \ (m,y) \in P_0\big\}. \end{aligned}$$

Definition

SSy(M) is the collection of all sets of the form

$$\{\bar{m} \in \omega : M \models \theta(\bar{m}, \bar{c})\},$$

where θ is an $\mathscr{L}_\mathrm{A}$ formula and $\bar{c} \in M$.

$(\omega, \mathrm{SSy}(M)) \models \mathrm{ACA}_0$

Proposition (Kaye–Kossak–W)

If $\omega \models \sigma$ and SSy(M) is closed under $(\cdot)'$, then $(M, \omega) \models \tau(\sigma)$.

Conclusion

Theorem (Kaye–Kossak–W)

If $M \models \mathrm{PA}$ such that $\mathrm{SSy}(M)$ is closed under $(\cdot)'$, then $0^{(\omega)}$ is parameter-free definable in (M, ω).

Intuition

The following properties are similar for $M \models \mathrm{PA}$.

- $M \succ \omega$.
- $\mathrm{SSy}(M)$ is closed under $(\cdot)'$.

Fact

If $M \succ \omega$ or $\mathrm{SSy}(M)$ is closed under $(\cdot)'$, then there is $b \in M$ such that

$$\omega < b < c$$

for all nonstandard definable elements $c \in M$.