On a Theorem of Seetapun

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Post's problem

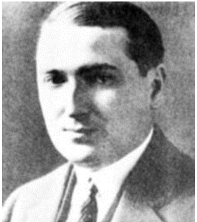
A Turing degree is c.e. if it contains a c.e. set.

The c.e. degrees form an upper-semi lattice with greatest and least elements, $\mathbf{0}'$ and $\mathbf{0}$.

Post:
Are there any other c.e. degrees?

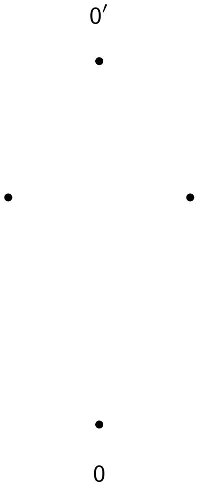
Post's efforts:

- simple sets,
- hypersimple sets,
- hyperhypersimple sets.



Friedberg-Muchnik Theorem

The answer to Post problem is “YES”.



Significance: a technique, called “priority injury argument”, was invented.

Sacks' Two Theorems

- Sacks Splitting Theorem: Every nonzero c.e. degree is the join of two incomparable c.e. degrees.
- Sacks Density Theorem: The c.e. degrees are dense.

New features of Sacks' theorems: complexity of injury arguments

Shoenfield conjectured that the structure of c.e. degrees is not complicated.

Shoenfield Conjecture

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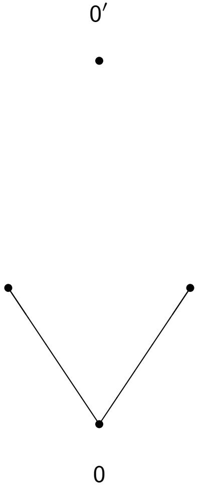
As an upper-semi lattice, the structure of c.e. degrees is countably categorical.

- If the conjecture is true, the theory of this countable structure is decidable.
- Lachlan and Yates first proved the existence of minimal pairs and hence Shoenfield Conjecture is wrong.

Minimal pairs, cappable degrees, noncapable degrees



High/Low hierarchy

- Jump operator - $A' = \{e : \Phi_e^A(e) \text{ converges}\}$.

$0'$

$\bullet$

high

high_2

high_3

low_3

low_2

low

$\bullet$

0

A decomposition of c.e. degrees

Theorem

Harrington's Work

- *Caps or Cups*
- *Caps and Cups*

A decomposition of c.e. degrees

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Harrington's Work

- *Caps or Cups*
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Theorem

AJSS's decomposition Theorem

- *All cappable degrees form an ideal of c.e. degrees;*
- *All noncappable degrees form a strong filter of c.e. degrees;*
- *A c.e. degree is noncappable if and only if it is low-cuppable;*

Cuppable degrees

Theorem

Recent Work

- *There is a low$_2$, but not low-cuppable, degree.* *(LWZ)*

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Cuppable degrees

Theorem

Recent Work

- *There is a low$_2$, but not low-cuppable, degree.* *(LWZ)*
- *There exists a cuppable degree, which is only high-cuppable to $\mathbf{0}'$.* *(GNW)*
- *There exists two cuppable degrees such that no incomplete c.e. degree can cup both to $\mathbf{0}'$ simultaneously. This implies that the quotient structure $\mathbf{R}/NCup$ contains a minimal pair.* *(LWY)*

Locally noncappable degrees

Definition: (Seetapun)

A nonzero c.e. degree $\mathbf{a}$ is locally noncappable if there is a c.e. degree $\mathbf{c}$ above $\mathbf{a}$ such that no nonzero c.e. degree below $\mathbf{c}$ can form a minimal pair with $\mathbf{a}$.

We say that $\mathbf{c}$ witnesses that $\mathbf{a}$ is locally noncappable.

Theorem: Downey, Stob

Any nonzero c.e. degree $\mathbf{a}$ bounds a nonzero c.e. degree $\mathbf{c}$ such that $\mathbf{c}$ is noncappable below $\mathbf{a}$.

Theorem: (Seetapun, 1991)

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Theorem: (Stephan and Wu)

The witness **c** can be high$_2$.

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- There is a high_2 degree bounding no bases of Slaman triples [Leonardi, 1996].

High permitting

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Every high c.e. degree bounds

- a minimal pair (Cooper, 1973);
- a high noncuppable degree (Harrington, around 1973);
- a Slaman triple (Shore and Slaman, 1993).

Thanks!