Ramsey's Theorem for Pairs and Reverse Mathematics

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Ramsey's Theorem

Definition

For $A \subseteq \mathbb{N}$, let $[A]^n$ denote the set of all n-element subsets of A.

Theorem (Ramsey, 1930)

Suppose $f : [\mathbb{N}]^n \to \{0, 1, \ldots, k-1\}$. Then there is an infinite set $H \subseteq \mathbb{N}$ f is constant on $[H]^n$.

H is called *f-homogeneous*.

Notation: Fix n and k, the particular version above is denoted by RT^n_k.

Motivations

- Informal reading: Within some sufficiently large systems, however disordered, there must be some order.
- Question: How complicated is the homogenous set H?
- Question: What information does H carry? E.g. does this infinite set tell us more about finite sets?
- (What are the consequences/strength of Ramsey's Theorem as a combinatorial principle?)
- Precise formulation requires some definitions from Recursion Theory and Reverse Mathematics.

Arithmetical Hierarchy

- Language of first order Peano Arithmetic: 0, S, $+$, $\times$; variables and quantifier are intended for individuals.
- Each formula are classified by the number of alternating blocks of quantifiers: Σ^0_n, Π^0_n and Δ^0_n formulas.
- Definable sets are classified by their defining formulas.
- Slogan: "Definability is computability": Recursive=Δ_1, and recursively enumerable sets = Σ_1 sets etc.

Fragments of First Order Peano Arithmetic

- Let $I\Sigma_n$ denote the induction schema for Σ^0_n-formulas; and $B\Sigma_n$ denote the Bounding Principle for Σ^0_n formulas.
- (Kirby and Paris, 1977) $\cdots \Rightarrow I\Sigma_{n+1} \Rightarrow B\Sigma_{n+1} \Rightarrow I\Sigma_n \Rightarrow \dots$
- (Slaman 2004) $I\Delta_n \Leftrightarrow B\Sigma_n$.

Fragments of Second Order Arithmetic

- Two sorted language: (first order part) + variables and quantifiers for sets.

- RCA_0: Σ^0_1-induction and Δ^0_1-comprehension:
  For $\varphi \in \Delta_1$, $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$.

- WKL_0: RCA_0 and every infinite binary tree has an infinite path.

- ACA_0: RCA_0 and for φ arithmetic, $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$.

- (ATR_0 and Π^1_1-CA_0.) Π^1_1-formulas are of the form $\forall X \varphi$ where φ is an arithmetic formula (with parameters).

Remarks on Axioms

- They all assert the existence of certain sets.
- Some are measured by syntactical complexity, e.g. ACA_0.
- Some are from the analysis of mathematical tools, e.g. WKL_0 corresponds to Compactness Theorem.

Basic Models

- A *model* $\mathcal{M}$ *of second-order arithmetic* consists $(M, 0, S, +, \times, \mathcal{S})$ where $(M, 0, S, +, \times)$ is its first-order part and the set variables are interpreted as members of $\mathcal{S}$.
- Models of RCA_0: Closure under $\leq_T$ and Turing join.
- In the (minimal) model of RCA_0, $\mathcal{S}$ only consists of $\mathcal{M}$-recursive sets.
- RCA_0 is the place to do constructive/finitary mathematics.

Remarks on Goals of Reversion

- Goal of Reverse Mathematics: What set existence axioms are needed to prove the theorems of ordinary, classical (countable) mathematics?
- Goal of Reverse Recursion Theory: What amount of induction are needed to prove the theorems of Recursion Theory, in particular, theorem about r.e. degrees.
- Motivation: To achieve these goals, we have to discover new proofs.

Rephrasing the motivating questions

- Question: Suppose f is recursive. How about the arithmetical complexity of the least complicated homogeneous set H?
- Question: Which system in Reverse Mathematics does Ramsey's Theorem correspond?
- (What are the first-order and second order consequences of Ramsey's Theorem?)

Some Earlier Results: (I)

Theorem (Jockusch, 1972)

1. *Every recursive colouring f has a Π^0_2 homogenous set H.*
2. *There is a recursive $f : [M]^3 \to \{0, 1\}$ all of whose homogenous set computes $0'$.*
3. *There is a recursive colouring of pairs which has no Σ^0_2 homogenous set.*

Corollary

Over RCA_0,

$$\mathrm{ACA}_0 \Leftrightarrow \mathrm{RT}^3_2 \Leftrightarrow \mathrm{RT}.$$

$$\mathrm{ACA}_0 \Rightarrow \mathrm{RT}^2_2 \quad \textit{and} \quad \mathrm{WKL}_0 \not\Rightarrow \mathrm{RT}^2_2.$$

Some Earlier Results: (II)

Theorem (Hirst 1987)

Over RCA_0*,*

$$\mathsf{RT}^2_2 \Rightarrow B\Sigma_2.$$

(This tells us the lower bound of its first order strength.)

Theorem (Seetapun and Slaman 1995)

There is an ideal J in the Turing degrees as follows.

- $0' \notin J$
- *For every* $f : [M]^2 \to \{0,1\}$ *in J, there is an infinite f-homogeneous H in J.*

Corollary

Over RCA_0*,*

$$\mathsf{ACA}_0 \Rightarrow \mathsf{RT}^2_2 \quad \textit{and} \quad \mathsf{RT}^2_2 \not\Rightarrow \mathsf{ACA}_0.$$

Some Earlier Results: (III)

- $f : [M]^2 \to \{0, 1\}$ is a called a *stable colouring* if for any x, $\lim_y f(x, y)$ exists.
- Stable Ramsey's Theorem for Pairs SRT^2_2 says homogenous sets exists for stable colourings.
- SRT^2_2 is equivalent to "For every Δ^0_2 property A, there is an infinite set H contained in or disjoint from A."

Theorem (Cholak, Jockusch and Slaman, 2001)

Over RCA_0*,*

$$\mathrm{RT}^2_2 \Leftrightarrow \mathrm{SRT}^2_2 + \mathrm{COH}.$$

(COH is another second order combinatorial principle.)

Conservation Results

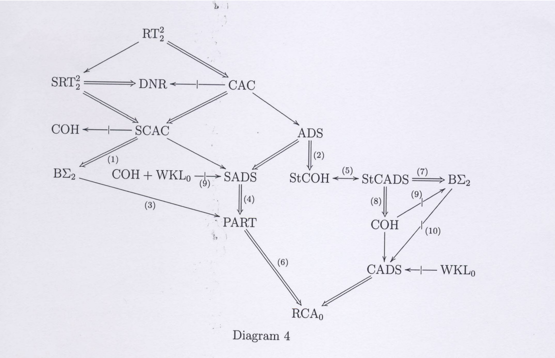
- Harrington observed that WKL_0 is Π^1_1-conservative over RCA_0. i.e., any Π^1_1-statement that is provable in WKL_0 is already provable in the system RCA_0.
- Conservation results are used to measure the weakness of the strength of a theorem.

Theorem (Cholak, Jockusch and Slaman 2001)

RT^2_2 *is* Π^1_1*-conservative over* $\mathsf{RCA}_0 + I\Sigma_2$.

Combinatorics below RT^2_2

Hirschfeldt and Shore [2007], *Combinatorial principles weaker than Ramsey's theorem for pairs*.



Some Resent Results

Theorem (Jiayi Liu, 2011)

Over RCA_0*,*

$$\mathsf{RT}^2_2 \not\Rightarrow \mathsf{WKL}_0.$$

Theorem (Chong, Slaman and Yang, 2011)

Over RCA_0*,* COH *is* Π^1_1*-conservative over* $\mathsf{RCA}_0 + B\Sigma_2$*.*

Remaining Questions and Obstacles

- Question 1: Over RCA_0, does SRT^2_2 imply RT^2_2?
- Question 2: Does SRT^2_2 imply $I\Sigma_2$? How about RT^2_2?
- Attempt for Q 1: Show that stable colourings always have a low homogenous sets. Or equivalently, every Δ^0_2-set contains or is disjoint from an infinite low set.

Theorem (Downey, Hirschfeldt, Lempp and Solomon, 2001)

There is a Δ^0_2 set with no infinite low subset in either it or its complement.

Nonstandard Approach

Chong (2005): We should look at nonstandard fragments of arithmetic, because:

- DFLS theorem is done on ω, whose proof involves infinite injury method thus requires $I\Sigma_2$.
- There is a model of $B\Sigma_2$ but not $I\Sigma_2$ in which every incomplete Δ^0_2 set is low.

Theorem (Chong, Slaman and Yang, 2012)

Over RCA_0*,*

$$\mathrm{SRT}^2_2 \not\Rightarrow \mathrm{RT}^2_2$$

$$\mathrm{SRT}^2_2 \not\Rightarrow I\Sigma_2.$$

Technical Remarks

- The first order part of the model satisfies $PA^{-} + B\Sigma_2^0$ but not $I\Sigma_2^0$.
- Also assumed
  - ω is the Σ_2^0-cut;
  - Σ_1^0-reflection property (and other conditions);
  - certain amount of saturation (to have sufficient codes).
- All these nonstandard features are crucial in the proof. By DHLS, the method does not apply to ω.

Further Results and Questions

- Theorem (to appear): RT^2_2 does not prove $I\Sigma^0_2$.
- Question: What happens in ω-model? Kind of “provability vs. truth” question.
- How about conservation results?

References

1. Simpson, *Subsystems of Second-Order Arithmetic*, (second edition), ASL and CUP 2009.

2. Hirschfeldt and Shore, *Combinatorial principles weaker than Ramsey's theorem for pairs*, JSL, 2007.

3. Liu Jiayi, RT^2_2 *does not imply* WKL_0, JSL 2011.

4. Chong, Slaman and Yang, *The Metamathematics of Stable Ramsey's Theorem for Pairs*, preprint.