Several versions of Friedman's self-embedding theorem

Keita Yokoyama

Tokyo Institute of Technology

February 20, 2013

Friedman's self-embedding theorem

Theorem (Friedman 1973)

If M is a countable model of PA*, there exists a self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$.*
($f(M)$ is a proper cut of M.)

Starting from Friedman's theorem, several different/precise versions of self-embedding theorems are studied.

- Ali Enayat "From Friedman to Tanaka, and beyond" CUNY Logic seminar, May 2012.

Friedman's self-embedding theorem

Theorem (Friedman 1973)

If M is a countable model of PA, *there exists a self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$.*
($f(M)$ is a proper cut of M.)

Starting from Friedman's theorem, several different/precise versions of self-embedding theorems are studied.

- Ali Enayat "From Friedman to Tanaka, and beyond" CUNY Logic seminar, May 2012.

Outline

1. "From Friedman": self-embedding for first-order arithmetic
   - Optimal self-embedding theorem for $\mathrm{I}\Sigma_n$
   - Some more variations
2. "To Tanaka": self-embedding for second-order arithmetic
   - Self-embedding theorems for subsystems of second-order arithmetic
   - Nonstandard arithmetic

Outline

1. "From Friedman": self-embedding for first-order arithmetic
   - Optimal self-embedding theorem for $\mathrm{I}\Sigma_n$
   - Some more variations
2. "To Tanaka": self-embedding for second-order arithmetic
   - Self-embedding theorems for subsystems of second-order arithmetic
   - Nonstandard arithmetic

Fragments of PA (Induction and bounding)

- Σ_n-induction: for $\varphi \in \Sigma_n$,

$$\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1)) \to \forall n \varphi(n).$$

- Σ_n-bounding: for $\varphi \in \Sigma_n$,

$$\forall u \exists v[\forall n \leq u \exists m\, \varphi(n,m) \to \forall n \leq u \exists m \leq v\, \varphi(n,m)].$$

$$\mathrm{I}\Sigma_n := \text{"basic axioms"} + \Sigma_n\text{-induction.}$$
$$\mathrm{B}\Sigma_n := \text{"basic axioms"} + \Sigma_n\text{-bounding.}$$
$$\mathrm{PA} := \bigcup_{n\in\omega} \mathrm{I}\Sigma_n = \bigcup_{n\in\omega} \mathrm{B}\Sigma_n.$$

Theorem (Paris, etc.)

$$\mathrm{I}\Sigma_0 < \mathrm{B}\Sigma_1 < \mathrm{I}\Sigma_1 < \mathrm{B}\Sigma_2 < \mathrm{I}\Sigma_2 < \mathrm{B}\Sigma_3 < \cdots < \mathrm{PA}.$$

Friedman's self-embedding theorem

Original self-embedding theorem.

Theorem (Friedman 1973)

If M is a countable model of PA, there exists a self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$.
($f(M)$ is a proper cut of M.)

We can sharpen this theorem as follows.

Theorem (In Kaye's book (1991))

Let $n \geq 0$.
If M is a countable model of $\mathrm{I}\Sigma_{n+1}$, there exists a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$.

Friedman's self-embedding theorem

Original self-embedding theorem.

Theorem (Friedman 1973)

If M is a countable model of PA*, there exists a self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$.*
($f(M)$ is a proper cut of M.)

We can sharpen this theorem as follows.

Theorem (In Kaye's book (1991))

Let $n \geq 0$.
If M is a countable model of $\mathrm{I}\Sigma_{n+1}$, there exists a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$.

Friedman's self-embedding theorem

Question

If a countable model M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$, then, is M a model of $\mathrm{I}\Sigma_{n+1}$?

No!

We can show that $M \models \mathrm{B}\Sigma_{n+1}$, but,

Proposition (folklore)

Let M be a countable recursively saturated model of $\mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *M is a model of $\mathrm{B}\Sigma_{n+1}$.*
- *M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$.*

$\Rightarrow$ There should be a "stronger" self-embedding theorem for $\mathrm{I}\Sigma_n$.

Friedman's self-embedding theorem

Question

If a countable model M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$, then, is M a model of $\mathrm{I}\Sigma_{n+1}$?

No!

We can show that $M \models \mathrm{B}\Sigma_{n+1}$, but,

Proposition (folklore)

Let M be a countable recursively saturated model of $\mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *M is a model of $\mathrm{B}\Sigma_{n+1}$.*
- *M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$.*

$\Rightarrow$ There should be a "stronger" self-embedding theorem for $\mathrm{I}\Sigma_n$.

Friedman's self-embedding theorem

Question

If a countable model M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$, then, is M a model of $\mathrm{I}\Sigma_{n+1}$?

No!

We can show that $M \models \mathrm{B}\Sigma_{n+1}$, but,

Proposition (folklore)

Let M be a countable recursively saturated model of $\mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *M is a model of $\mathrm{B}\Sigma_{n+1}$.*
- *M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$.*

$\Rightarrow$ There should be a "stronger" self-embedding theorem for $\mathrm{I}\Sigma_n$.

Friedman's self-embedding theorem

Question

If a countable model M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$, then, is M a model of $\mathrm{I}\Sigma_{n+1}$?

No!

We can show that $M \models \mathrm{B}\Sigma_{n+1}$, but,

Proposition (folklore)

Let M be a countable recursively saturated model of $\mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *M is a model of $\mathrm{B}\Sigma_{n+1}$.*
- *M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$.*

$\Rightarrow$ There should be a "stronger" self-embedding theorem for $\mathrm{I}\Sigma_n$.

Friedman's self-embedding theorem

Question

If a countable model M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$, then, is M a model of $\mathrm{I}\Sigma_{n+1}$?

No!

We can show that $M \models \mathrm{B}\Sigma_{n+1}$, but,

Proposition (folklore)

Let M be a countable recursively saturated model of $\mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *M is a model of $\mathrm{B}\Sigma_{n+1}$.*
- *M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$.*

$\Rightarrow$ There should be a "stronger" self-embedding theorem for $\mathrm{I}\Sigma_n$.

Self-embedding theorem for $\mathrm{I}\Sigma_n$

We can add an extra condition for $f(M)$.

Definition

Let M be a model of $\mathrm{I}\Sigma_0$, and $I \subsetneq_e M$.
Then, I is said to be semi-regular if for any M-finite set $X \subseteq M$,

$$|X| \in I \Rightarrow X \cap I \text{ is bounded in } I.$$

Note that semi-regular cut is very useful in the model theory of arithmetic. (It is called "finiteness" in nonstandard arithmetic.)

Theorem

Let M be a countable model of $\mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *M is a model of $\mathrm{I}\Sigma_{n+1}$.*
- *M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$ and $f(M)$ is semi-regular.*

Self-embedding theorem for $\mathrm{I}\Sigma_n$

We can add an extra condition for $f(M)$.

Definition

Let M be a model of $\mathrm{I}\Sigma_0$, and $I \subsetneq_e M$.
Then, I is said to be semi-regular if for any M-finite set $X \subseteq M$,

$$|X| \in I \Rightarrow X \cap I \text{ is bounded in } I.$$

Note that semi-regular cut is very useful in the model theory of arithmetic. (It is called "finiteness" in nonstandard arithmetic.)

Theorem

Let M be a countable model of $\mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *M is a model of $\mathrm{I}\Sigma_{n+1}$.*
- *M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$ and $f(M)$ is semi-regular.*

Self-embedding theorem for $\mathrm{I}\Sigma_n$

We can add an extra condition for $f(M)$.

Definition

Let M be a model of $\mathrm{I}\Sigma_0$, and $I \subsetneq_e M$.
Then, I is said to be semi-regular if for any M-finite set $X \subseteq M$,

$$|X| \in I \Rightarrow X \cap I \text{ is bounded in } I.$$

Note that semi-regular cut is very useful in the model theory of arithmetic. (It is called "finiteness" in nonstandard arithmetic.)

Theorem

Let M be a countable model of $\mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *M is a model of $\mathrm{I}\Sigma_{n+1}$.*
- *M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$ and $f(M)$ is semi-regular.*

Self-embedding theorem for $\mathrm{I}\Sigma_n$

We can add an extra condition for $f(M)$.

Definition

Let M be a model of $\mathrm{I}\Sigma_0$, and $I \subsetneq_e M$.
Then, I is said to be semi-regular if for any M-finite set $X \subseteq M$,

$$|X| \in I \Rightarrow X \cap I \text{ is bounded in } I.$$

Note that semi-regular cut is very useful in the model theory of arithmetic. (It is called "finiteness" in nonstandard arithmetic.)

Theorem

Let M be a countable model of $\mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *M is a model of $\mathrm{I}\Sigma_{n+1}$.*
- *M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$ and $f(M)$ is semi-regular.*

Q-semi-regular embedding

Fix a Σ^0_0-definable $Q : [M]^{<M} \to M$ such that $X \subseteq Y \Rightarrow Q(X) \leq Q(Y)$.

Definition

- $I \subsetneq_e M$ is said to be Q-semi-regular if for any $X \subseteq_{\mathrm{fin}} M$,

$$Q(X) \in I \Rightarrow X \cap I \text{ is bounded in } I.$$

- For any definable $X \subseteq M$,

$$Q(X) = \sup\{Q(X \cap [0, a]) \mid a \in M\}.$$

Theorem

Let $M \models \mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *$M \models$ "$Q(X) < \infty \leftrightarrow X$ is bounded" for any Σ_1-definable $X \subseteq M$.*
- *M has a Q-semi-regular self-embedding.*

Q-semi-regular embedding

Fix a Σ^0_0-definable $Q : [M]^{<M} \to M$ such that $X \subseteq Y \Rightarrow Q(X) \leq Q(Y)$.

Definition

- $I \subsetneq_e M$ is said to be Q-semi-regular if for any $X \subseteq_{\mathrm{fin}} M$,

$$Q(X) \in I \Rightarrow X \cap I \text{ is bounded in } I.$$

- For any definable $X \subseteq M$,

$$Q(X) = \sup\{Q(X \cap [0, a]) \mid a \in M\}.$$

Theorem

Let $M \models \mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *$M \models$ "$Q(X) < \infty \leftrightarrow X$ is bounded" for any Σ_1-definable $X \subseteq M$.*
- *M has a Q-semi-regular self-embedding.*

Q-semi-regular embedding

Fix a Σ^0_0-definable $Q : [M]^{<M} \to M$ such that $X \subseteq Y \Rightarrow Q(X) \leq Q(Y)$.

Corollary

Let

$$(M, S) \models \mathrm{RCA}_0 + \text{“}\forall X\ Q(X) < \infty \leftrightarrow X \text{ is bounded”.}$$

Then, there exists $S' \supseteq S$ such that

$$(M, S') \models \mathrm{WKL}_0 + \text{“}\forall X\ Q(X) < \infty \leftrightarrow X \text{ is bounded”.}$$

Thus, for any Π^1_1-formula φ,

$$\mathrm{RCA}_0 + \text{“}\forall X\ Q(X) < \infty \leftrightarrow X \text{ is bounded”} \vdash \varphi$$

if and only if

$$\mathrm{WKL}_0 + \text{“}\forall X\ Q(X) < \infty \leftrightarrow X \text{ is bounded”} \vdash \varphi.$$

Self-embedding theorem for PA

Definition

Let M be a model of Iz, and $I \subsetneq_e M$.
Then, I is said to be strong if for any $b \in M \setminus I$ and M-finite sequence $\langle c_i \in M \mid i < b \rangle$,

$$\exists d \in M (c_i < d \Leftrightarrow c_i \in I).$$

Theorem (folklore)

Let M be a countable model of $\mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *M is a model of* PA.
- *M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$ and $f(M)$ is strong.*

When there is a self-embedding between $a < b$?

Theorem (In Kaye's book)

Let M be a countable model of $\mathrm{I}\Sigma_{n+1}$, and let $a, b \in M$. If for any Σ_n-definable partial function $\nu : \subseteq M \to M$, $\nu(a) < b$, then there exists a Σ_n-elementary self-embedding $f : M \to M$ such that $a \in f(M) \subsetneq_e M$ and $b \notin f(M)$.

However, this condition was not optimal for the above question. Recently, an optimal version is proved.

Theorem (Enayat 201X)

Let M be a countable model of $\mathrm{I}\Sigma_{n+1}$, and let $a, b \in M$. Then the following are equivalent.

- *For any Σ_n-definable total function $\nu : M \to M$, $\nu(a) < b$.*
- *There exists a Σ_n-elementary self-embedding $f : M \to M$ such that $a \in f(M) \subsetneq_e M$ and $b \notin f(M)$.*

When there is a self-embedding between $a < b$?

Theorem (In Kaye's book)

Let M be a countable model of $\mathrm{I}\Sigma_{n+1}$, and let $a, b \in M$. If for any Σ_n-definable partial function $\nu : \subseteq M \to M$, $\nu(a) < b$, then there exists a Σ_n-elementary self-embedding $f : M \to M$ such that $a \in f(M) \subsetneq_e M$ and $b \notin f(M)$.

However, this condition was not optimal for the above question.

Recently, an optimal version is proved.

Theorem (Enayat 201X)

Let M be a countable model of $\mathrm{I}\Sigma_{n+1}$, and let $a, b \in M$. Then the following are equivalent.

- *For any Σ_n-definable total function $\nu : M \to M$, $\nu(a) < b$.*
- *There exists a Σ_n-elementary self-embedding $f : M \to M$ such that $a \in f(M) \subsetneq_e M$ and $b \notin f(M)$.*

When there is a self-embedding between $a < b$?

Theorem (In Kaye's book)

Let M be a countable model of $\mathrm{I}\Sigma_{n+1}$, and let $a, b \in M$. If for any Σ_n-definable partial function $\nu : \subseteq M \to M$, $\nu(a) < b$, then there exists a Σ_n-elementary self-embedding $f : M \to M$ such that $a \in f(M) \subsetneq_e M$ and $b \notin f(M)$.

However, this condition was not optimal for the above question. Recently, an optimal version is proved.

Theorem (Enayat 201X)

Let M be a countable model of $\mathrm{I}\Sigma_{n+1}$, and let $a, b \in M$. Then the following are equivalent.

- *For any Σ_n-definable total function $\nu : M \to M$, $\nu(a) < b$.*
- *There exists a Σ_n-elementary self-embedding $f : M \to M$ such that $a \in f(M) \subsetneq_e M$ and $b \notin f(M)$.*

Self-embedding theorem for $\mathrm{I}\Sigma_n$

Here is another characterization of $\mathrm{I}\Sigma_n$.

Theorem

Let M be a countable model of $\mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *M is a model of $\mathrm{I}\Sigma_{n+1}$.*
- *For any $a \in M$, M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$ and $f(x) = x$ for any $x < a$.*

Self-embedding theorem for $\mathrm{I}\Sigma_n$

Here is another characterization of $\mathrm{I}\Sigma_n$.

Theorem

Let M be a countable model of $\mathrm{I}\Sigma_0$. Then, the following are equivalent.

- *M is a model of $\mathrm{I}\Sigma_{n+1}$.*
- *For any $a \in M$, M has a Σ_n-elementary self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$ and $f(x) = x$ for any $x < a$.*

Outline

1. "From Friedman": self-embedding for first-order arithmetic
   - Optimal self-embedding theorem for $\mathrm{I}\Sigma_n$
   - Some more variations
2. "To Tanaka": self-embedding for second-order arithmetic
   - Self-embedding theorems for subsystems of second-order arithmetic
   - Nonstandard arithmetic

Subsystems of second-order arithmetic

Big five plus one

- RCA_0: "discrete ordered semi-ring"+Σ^0_1 induction
  +Δ^0_1 comprehension.
- WWKL_0: RCA_0 + weak weak König's lemma.
  (Any tree which has a positive measure has a path.)
- WKL_0: RCA_0 + weak König's lemma.
- ACA_0: RCA_0 + Σ^1_0 comprehension.
- ATR_0: RCA_0 + arithmetical transfinite recursion.
- $\Pi^1_1\mathsf{CA}_0$: RCA_0 + Π^1_1 comprehension.

Self-embedding theorem for second-order arithmetic

'For second-order arithmetic, Tanaka's theorem is the "only and definitive" version.' (Enayat)

Theorem (Tanaka)

Let (M, S) be a countable model of RCA_0. Then, the following are equivalent.

- *(M, S) is a model of WKL_0.*
- *There exists a self-embedding $f : (M, S) \to (M, S)$ such that $f(M) \subsetneq_e M$ and $f(S) \upharpoonright f(M) = S \upharpoonright f(M)$.*

($S \upharpoonright I = \{X \cap I \mid X \in S\}$.)

Self-embedding theorem for second-order arithmetic

Theorem (Wong's note)

Let (M, S) be a countable recursively saturated model of RCA_0^*. Then, the following are equivalent.

- *(M, S) is a model of WKL_0^*.*
- *There exists a self-embedding $f : (M, S) \to (M, S)$ such that $f(M) \subsetneq_e M$ and $f(S) \upharpoonright f(M) = S \upharpoonright f(M)$.*

($S \upharpoonright I = \{X \cap I \mid X \in S\}$.)

Self-embedding theorem for second-order arithmetic

With the notion of strong cut or elementarity, we can characterize ACA_0.

Corollary

Let (M, S) be a countable model of RCA_0, and let $n \geq 1$. Then, the following are equivalent.

- *(M, S) is a model of ACA_0.*
- *There exists a self-embedding $f : (M, S) \to (M, S)$ such that $f(M) \subsetneq_e M$, $f(M)$ is strong and $f(S) \restriction f(M) = S \restriction f(M)$.*
- *There exists a Σ^0_n-elementary self-embedding $f : (M, S) \to (M, S)$ such that $f(M) \subsetneq_e M$ and $f(S) \restriction f(M) = S \restriction f(M)$.*

However, I want some more versions!

Self-embedding theorem for second-order arithmetic

With the notion of strong cut or elementarity, we can characterize ACA_0.

Corollary

Let (M, S) be a countable model of RCA_0, and let $n \geq 1$. Then, the following are equivalent.

- *(M, S) is a model of ACA_0.*
- *There exists a self-embedding $f : (M, S) \to (M, S)$ such that $f(M) \subsetneq_e M$, $f(M)$ is strong and $f(S) \upharpoonright f(M) = S \upharpoonright f(M)$.*
- *There exists a Σ^0_n-elementary self-embedding $f : (M, S) \to (M, S)$ such that $f(M) \subsetneq_e M$ and $f(S) \upharpoonright f(M) = S \upharpoonright f(M)$.*

However, I want some more versions!

M-finite complexity

Definition (M-finite complexity)

Let M be a model of $\mathrm{I}\Sigma_0$.

An *M-finite complexity* is an M-finite partial function $k :\subseteq 2^{<M} \to M$ such that $\sum_{\sigma\in\mathrm{dom}(k)} 2^{-k(\sigma)} \leq 1$. We define $k(\sigma) = \infty$ if $\sigma \notin \mathrm{dom}(k)$.

Definition

Let (M, S) be a model of RCA_0 and N be a model of $\mathrm{I}\Sigma_0$ such that $M \subsetneq_e N$.

- $(M, S) \subseteq_{e,r} N$ if for any finite complexity $k \in N$ there exists a set $X \in S$ and $c \in M$ such that $\forall n \in M\ k(X \restriction n) > n - c$.
- $(M, S) \subseteq_{e,d} N$ if for any finite complexity $k \in N$ there exists a set $f, X \in S$ such that $\forall n \in M\ k(X \restriction n) > f(n)$.

Self-embedding theorem for second-order arithmetic

Theorem

Let (M, S) be a countable model of RCA_0. Then, the following are equivalent.

1. *(M, S) is a model of WWKL_0.*
2. *There exists a self-embedding $f : (M, S) \to (M, S)$ such that $(f(M), f(S) \upharpoonright f(M)) \subseteq_{e,r} M$.*

Theorem

Let (M, S) be a countable model of RCA_0. Then, the following are equivalent.

1. *(M, S) is a model of $\mathrm{RCA}_0 + \mathrm{DNR}$.*
2. *There exists a self-embedding $f : (M, S) \to (M, S)$ such that $(f(M), f(S) \upharpoonright f(M)) \subseteq_{e,d} M$.*

With elementarity

Theorem (Avigad, Dean, Rute)

n-WWKL_0 consists of RCA_0 plus the following assertion:

> *any Δ^0_n-definable tree which has a positive measure has a path.*

Note that 2-WWKL_0 is equivalent to the Lebesgue convergence theorem. (A.D.R.)

Theorem

Let (M, S) be a recursively saturated countable model of RCA_0, and let $n \geq 1$. Then, the following are equivalent.

1. *(M, S) is a model of WWKL_0.*
2. *There exists a Σ^0_{n-1}-elementary self-embedding $f : (M, S) \to (M, S)$ such that $(f(M), f(S) \upharpoonright f(M)) \subseteq_{e,r} M$.*

Self-embedding theorem for second-order arithmetic

With the stronger notion of cut, we have the following.

Theorem

Let (M, S) be a countable model of RCA_0. Then, the following are equivalent.

1. *(M, S) is a model of Π^1_1-CA_0.*
2. *There exists a self-embedding $f : (M, S) \to (M, S)$ such that $f(M) \subsetneq_e M$ is a Ramsey strong cut and $f(S) \upharpoonright f(M) = S \upharpoonright f(M)$.*

Nonstandard arithmetic

Tanaka's idea:

- Self-embedding theorem is very useful to construct a "good end-extension" to do nonstandard analysis within weak arithmetic.

Theorem (nonstandard arithmetic)

1. *The system consists of* STP *(standard part principle) and* Σ^0_1*-overspill is a conservative extension of* WKL_0*. (Tanaka)*
2. *The system consists of* STP *(standard part principle) and* Σ^1_0*-transfer principle is a conservative extension of* ACA_0*.*
3. *The system consists of* LMP *(for nonstandard measure theory),* Σ^0_1*-overspill and* Σ^0_{n-1}*-transfer principle is a conservative extension of* n-WWKL_0*. (Simpson/Y)*

Open questions

- Find self-embedding theorems which can characterize ATR_0, RT^2_2, etc.
- Show stronger conservation results for nonstandard arithmetic by using self-embedding theorems.
- What is the relation between self-embedding theorems and saturation principles?

Open questions

- Find self-embedding theorems which can characterize ATR_0, RT^2_2, etc.
- Show stronger conservation results for nonstandard arithmetic by using self-embedding theorems.
- What is the relation between self-embedding theorems and saturation principles?

Open questions

- Find self-embedding theorems which can characterize ATR_0, RT^2_2, etc.
- Show stronger conservation results for nonstandard arithmetic by using self-embedding theorems.
- What is the relation between self-embedding theorems and saturation principles?

Open questions

- Find self-embedding theorems which can characterize ATR_0, RT^2_2, etc.
- Show stronger conservation results for nonstandard arithmetic by using self-embedding theorems.
- What is the relation between self-embedding theorems and saturation principles?

Thank you!