A Categorical Description of Relativization

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Outline

Objective

Concept

Non-Computability in Categories

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Non-Computability in Categories

How to deal with non-computability in computable analysis?
$-->$ Relativizations to oracles (computability with oracles)

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Concept

Non-Computability in Categories

How to deal with non-computability in computable analysis?
$-->$ Relativizations to oracles (computability with oracles)

Objective

To give a categorical description of "relativization to oracles"

Goal

We propose to reformulate the following proposition
on a categorical setting

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on a categorical setting

Proposition

For a given represented space (X, δ_X), if δ_X is addmissible, then

oracle co-r.e. closedness coincides with topological closedness

for every subset of X

Preliminaries on TTE

Type-2 Theory of Effectivity

- A framework of computable analysis
- It provides us "de facto standard" terminologies

Preliminaries on TTE: 1/5

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- (Type-2) Computability is defined for partial functions on Cantor space
- Oracle computability is also defined

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Represented Space

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Represented Space

- a representation of a set X:

  a partial surjection from Cantor space to X

$$\begin{array}{c} X \\ \Big\uparrow{\scriptstyle \delta} \\ \mathrm{supp}(\delta) \subseteq 2^{\omega} \end{array}$$

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- a representation of a set X:

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$$\begin{array}{c} X \\ \uparrow\!\!\!\uparrow \delta \\ \mathrm{supp}(\delta) \subseteq 2^{\omega} \end{array}$$

- a represented space:

  a set equipped with a representation

Preliminaries on TTE: 2/5

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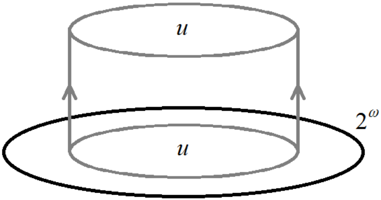
Example 1

Each $u \subseteq 2^\omega$ can be regarded as a represented space w.r.t. the representation δ_u defined as follows:

$$\delta_u(p) = \begin{cases} p & \text{if } p \in u \\ \text{undefined} & \text{otherwise} \end{cases}$$

where $p \in 2^\omega$



Preliminaries on TTE: 3/5

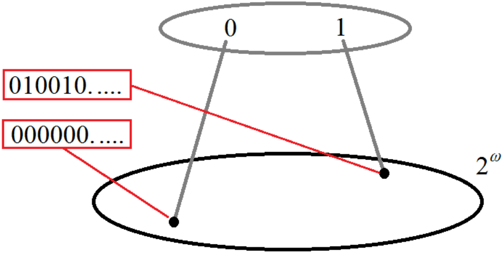
Preliminaries on TTE: 3/5

Example 2

We define a representation δ_Ω of 2 as follows:

$$\delta_\Omega(p) = \begin{cases} 0 & \text{if } p(i) = 0 \ ({}^\forall i \in \omega) \\ 1 & \text{otherwise} \end{cases}$$

where $p \in 2^\omega$



Preliminaries on TTE: 4/5

(X, δ_X), (Y, δ_Y): represented spaces

Preliminaries on TTE: 4/5

(X, δ_X), (Y, δ_Y): represented spaces

Relatively Computable Function

Each $f : X \to Y$ is said to be computable w.r.t. δ_X, δ_Y if there is a computable partial function g on 2^ω which makes the following diagram commute

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
\mathrm{supp}(\delta_X) \arrow[u, "\delta_X"] \arrow[r, "{}^{\exists}g"'] & \mathrm{supp}(\delta_Y) \arrow[u, "\delta_Y"']
\end{tikzcd}$$

Preliminaries on TTE: 4/5

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Oracle computability can also be extended in the same manner

Preliminaries on TTE: 5/5

(X, δ_X): represented space
u: a subset of X

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(X, δ_X): represented space
u: a subset of X

Co-r.e. Closedness

- We denote by $\mathrm{ch}_u : X \to 2$ its characteristic function i.e. the unique function such that $u = \mathrm{ch}_u^{-1}[\{0\}]$
- u is said to be (oracle) co-r.e. closed if ch_u is (oracle) computable w.r.t. δ_X, δ_Ω

Preliminaries on TTE: 5/5

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Topological Closedness

- One can think (X, δ_X) as a topological space w.r.t. the quotient topology induced from Cantor topology via δ_X
- u is said to be closed if it is closed w.r.t. the quotient topology

Preliminaries on Category Theory

We introduce:

- three examples of categories
- one example of functors
- the notion of factorization system

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Notations

$\mathbb{E}$	:	arbitrarily fixed category
$\mathrm{Iso}_{\mathbb{E}}$	:	the class of all isomorphisms
$\mathrm{Epi}_{\mathbb{E}}$	:	the class of all epimorphisms
$\mathrm{Mono}_{\mathbb{E}}$	:	the class of all monomorphisms

Preliminaries on Category Theory: 1/6

Example 1

`Set`

- object: small sets
- morphism: functions

Preliminaries on Category Theory: 1/6

Example 1

Set

- object: small sets
- morphism: functions

Example 2

Cp

- object: subsets of Cantor space
- morphism: computable total functions

Preliminaries on Category Theory: 1/6

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- object: small sets
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Example 2

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- object: subsets of Cantor space
- morphism: computable total functions

Example 3

Rep

- object: represented spaces
- morphism: computable total functions

Preliminaries on Category Theory: 2/6

A functor U from Cp to $\mathtt{Rep}$ can be defined as follows:

- object: $u \mapsto (u, \delta_u)$
- morphism: $g \mapsto g$

Preliminaries on Category Theory: 3/6

epi-mono factorizability of Set

For each morphism $X \xrightarrow{f} Y$ in $\mathtt{Set}$, there exists a pair of a epimorphism (surjective function) e and a monomorphism (injective function) m which makes the following diagram commute

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & & Y \\ & \cdot \arrow[ur, "m"'] & \end{tikzcd}$$

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Factorization System

- A factorization system $(\mathscr{S}, \mathscr{T})$ on $\mathbb{E}$ is defined as a pair of two classes of morphisms in $\mathbb{E}$
- A factorization system $(\mathscr{S}, \mathscr{T})$ is said to be proper if $\mathscr{S} \subseteq \mathrm{Epi}_{\mathbb{E}}$ and $\mathscr{T} \subseteq \mathrm{Mono}_{\mathbb{E}}$

Preliminaries on Category Theory: 4/6

Example: On Set

- $(\mathrm{Epi}_{\mathtt{Set}}, \mathrm{Mono}_{\mathtt{Set}})$ forms a proper factorization system on $\mathtt{Set}$
- this fact can be generalized to an arbitrary topos

Preliminaries on Category Theory: 4/6

Example: On $\mathtt{Set}$

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Example: On Cp

- $\mathscr{S}_{\mathrm{Cp}}$: the class of all surjective morphisms in Cp
- there is an uniquely determined class of morphisms $\mathscr{T}_{\mathrm{Cp}}$ s.t. $(\mathscr{S}_{\mathrm{Cp}}, \mathscr{T}_{\mathrm{Cp}})$ forms a proper factorization system on Cp
- all morphisms from $\mathscr{T}_{\mathrm{Cp}}$ are injective

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Example: On Rep

One can also define a proper factorization system $(\mathscr{S}_{\mathtt{Rep}}, \mathscr{T}_{\mathtt{Rep}})$ on $\mathtt{Rep}$ in the same manner with the case of Cp

Preliminaries on Category Theory: 5/6

$(\mathscr{S}, \mathscr{T})$: proper factorization system on $\mathbb{E}$

Preliminaries on Category Theory: 5/6

$(\mathscr{S}, \mathscr{T})$: proper factorization system on $\mathbb{E}$

Definition: Image

For each $X \xrightarrow{f} Y$ in $\mathbb{E}$ and each $(\cdot \xrightarrow{u} X) \in \mathscr{T}$, in the following factorization of fu

$$\begin{tikzcd} X \arrow[rr, "fu"] \arrow[dr, "s"'] & & Y \\ & \cdot \arrow[ur, "t"'] & \end{tikzcd}$$

we call t an image of u by f if $s \in \mathscr{S}$ and $t \in \mathscr{T}$

We usually denote by $f[u]$ an image of u by f

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We usually denote by $f[u]$ an image of u by f

Example: In $\mathtt{Set}$, $\mathtt{Cp}$ or $\mathtt{Rep}$

One can see the equality range$(f[u]) = f[$range$(u)]$

Preliminaries on Category Theory: 6/6

For each $(\cdot \xrightarrow{t} X), (\cdot \xrightarrow{t'} X) \in \mathrm{Mono}_{\mathbb{E}}$, we define:

$t \leq t' \iff$ there is a (necessarily unique) morphism j which makes the following triangle commute

$$\begin{tikzcd}
\cdot \arrow[rr, "j"] \arrow[dr, "t"'] & & \cdot \arrow[dl, "t'"] \\
& X &
\end{tikzcd}$$

Preliminaries on Category Theory: 6/6

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Example: In Set, Cp or Rep

$t \leq t'$ iff $\mathrm{range}(t) \subseteq \mathrm{range}(t')$

Fundamental Class

We introduce:

- our mathematical settings
- the notion of *fundamental class*

Notations

$\mathbb{E}$	:	finitely complete category
$(\mathscr{S}, \mathscr{T})$	:	proper factorization system on $\mathbb{E}$

Fundamental Class: 1/3

Assumptions

- $\mathscr{S}$ is stable under pullback

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  one has $s' \in \mathscr{S}$ whenever $s \in \mathscr{S}$

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- $\mathbb{E}$ has $\mathscr{T}$-intersection

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- $\mathbb{E}$ has $\mathscr{T}$-intersection
  i.e. if $\{(\cdot \xrightarrow{t_i} X)\}_{i \in I}$ is a family on $\mathscr{T}$, there exists $(\cdot \xrightarrow{t} X) \in \mathscr{T}$

  s.t. for each $(\cdot \xrightarrow{t'} X) \in \mathscr{T}$, $t' \leq t$ iff $t' \leq t_i$ $({}^{\forall} i \in I)$

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  s.t. for each $(\cdot \xrightarrow{t'} X) \in \mathscr{T}$, $t' \leq t$ iff $t' \leq t_i$ $({}^\forall i \in I)$

In the case of our examples `Set`, Cp and `Rep`, the above two assumptions are certainly hold

Fundamental Class: 2/3

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- Each fundamental class can be thought of as defining a topology-like structure on $\mathbb{E}$

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Definition

Each $\mathscr{F} \subseteq \mathscr{T}$ is said to be a fundamental class on $\mathbb{E}$ if:

- $\mathscr{F}$ contains all isomorphisms
- $\mathscr{F}$ is closed under composition
- $\mathscr{F}$ is stable under pullback

Fundamental Class: 3/3

Example: On $\mathtt{Set}$

Both $\mathrm{Iso}_{\mathtt{Set}}$ and $\mathrm{Mono}_{\mathtt{Set}}$ form fundamental classes

Fundamental Class: 3/3

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Both $\mathrm{Iso}_{\mathtt{Set}}$ and $\mathrm{Mono}_{\mathtt{Set}}$ form fundamental classes

Example: On Cp

We define a fundamental class $\Pi^0_{1,\mathrm{Cp}}$ on Cp as follows:

$$t \in \Pi^0_{1,\mathrm{Cp}} \iff \mathrm{range}(t) \text{ is co-r.e. closed in } u$$

where $(\cdot \xrightarrow{t} u) \in \mathscr{T}_{\mathrm{Cp}}$

Fundamental Class: 3/3

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Example: On Rep

We define a fundamental class $\Pi^0_{1,\mathrm{Rep}}$ on $\mathtt{Rep}$ as follows:

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Description

We give a description of each of:

- oracles
- relativization to oracles
- generation of topologies

Description: 1/3

Definition

Each $\alpha \in \mathbb{E}$ is said to be an imaginary

$$\text{if } (\alpha \xrightarrow{!} 1) \in \mathscr{S} \cap \mathrm{Mono}_{\mathbb{E}}$$

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Example: In Cp

Each $\alpha \in \mathrm{Cp}$ is an imaginary if and only if α is a singleton

i.e. it is being of the form $\alpha = \{*\}$ where $* \in 2^{\omega}$

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Example: In Rep

Each $(X, \delta_X) \in \mathsf{Rep}$ is an imaginary if and only if X is a singleton

Description: 2/3

- $[\mathscr{T}] = \{\mathscr{F} \subseteq \mathscr{T} : \mathscr{F}$ is a fundamental class on $\mathbb{E}\}$
- $[\mathscr{T}]$ can be regarded as a partially ordered system w.r.t. $\subseteq$

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We define a closure operator $\mathscr{I} : [\mathscr{T}] \to [\mathscr{T}]$ as follows

$$\mathscr{I}\mathscr{F} = \{t \in \mathscr{T} : {}^{\exists}\alpha\text{: imaginary s.t. } t \times \mathrm{id}_\alpha \in \mathscr{F}\}$$

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Example: On Cp

For every $(\cdot \xrightarrow{t} u) \in \mathscr{T}_{\mathrm{Cp}}$, the following equivalence hold:

$$t \in \mathscr{I}\Pi^0_{1,\mathrm{Cp}} \iff \text{range}(t) \text{ is oracle co-r.e. closed in } u$$

Description: 3/3

We define a closure operator $\mathscr{L} : [\mathscr{T}] \to [\mathscr{T}]$ as follows

$$\mathscr{L}\mathscr{F} = \bigcap\{\mathscr{F}' \in [\mathscr{T}] : \mathscr{F} \subseteq \mathscr{F}', \mathscr{F} \text{ is closed under } \mathscr{T}\text{-intersection}\}$$

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$$t \in \mathscr{L}\Pi^0_{1,\mathrm{Cp}} \iff \text{range}(t) \text{ is topologically closed in } u$$

Reformulate: Goal

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For a given represented space (X, δ_X), if δ_X is addmissible, then

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Question

Let $\mathscr{F}$ be a fundamental class on $\mathbb{E}$.

When does the equality $\mathscr{I}\mathscr{F} = \mathscr{L}\mathscr{F}$ hold?

Main Results

We introduce our two main results

- The first one: concerning the inclusion $\mathscr{I}\mathscr{F} \subseteq \mathscr{L}\mathscr{F}$
- The second one: concerning the equality $\mathscr{I}\mathscr{F} = \mathscr{L}\mathscr{F}$

The First One: 1/2

$\mathscr{F}$: fundamental class on $\mathbb{E}$

Definition

Each $X \xrightarrow{f} Y$ in $\mathbb{E}$ is said to be $\mathscr{F}$-closed
if for every $(\cdot \xrightarrow{u} X) \in \mathscr{F}$ its image $f[u]$ belongs to $\mathscr{F}$ again

The First One: 1/2

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Definition

Each $X \in \mathbb{E}$ is said to be $\mathscr{F}$-compact if the second projection
$X \times Y \xrightarrow{\pi_2} Y$ is always $\mathscr{F}$-closed for every $Y \in \mathbb{E}$

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Definition

Each $X \in \mathbb{E}$ is said to be $\mathscr{F}$-compact if the second projection $X \times Y \xrightarrow{\pi_2} Y$ is always $\mathscr{F}$-closed for every $Y \in \mathbb{E}$

One can give an alternative description of Heine-Borel compactness using the above generalized notion of compactness

The First One: 2/2

Theorem

If $\mathbb{E}$ is well-powered, then the following two conditions are equivalent:

(i) $\mathscr{I}\mathscr{F} \subseteq \mathscr{L}\mathscr{F}$;

(ii) all imaginaries are $\mathscr{L}\mathscr{F}$-compact.

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(i) $\mathscr{I}\mathscr{F} \subseteq \mathscr{L}\mathscr{F}$;

(ii) all imaginaries are $\mathscr{L}\mathscr{F}$-compact.

One can interpret as follows:

$\mathbb{E}$	$\mathscr{F}$
Cp	Π^0_1

The condition (ii), and thus also (i), is certainly fulfilled in this case

The Second One: 1/2

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A functor $G : \mathbb{E} \to \mathbb{E}'$ with certain properties is supposed to be given

The Second One: 1/2

A functor $G : \mathbb{E} \to \mathbb{E}'$ with certain properties is supposed to be given

Theorem

One has $\mathscr{I}\mathscr{F} = \mathscr{L}\mathscr{F}$ if the following three conditions hold:

(i) all imaginaries of $\mathbb{E}$ are $\mathscr{L}\mathscr{F}$-compact;

(ii) $\mathrm{id}_X \in {}^{G}\mathscr{I}\mathscr{F}$ for every $X \in \mathbb{E}$;

(iii) ${}^{G}\mathscr{I}\mathscr{F}$ is included in $\mathscr{I}\mathscr{F}$.

The Second One: 1/2

A functor $G : \mathbb{E} \to \mathbb{E}'$ with certain properties is supposed to be given

Theorem

One has $\mathscr{I}\mathscr{F} = \mathscr{L}\mathscr{F}$ if the following three conditions hold:

(i) all imaginaries of $\mathbb{E}$ are $\mathscr{L}\mathscr{F}$-compact;

(ii) $\mathrm{id}_X \in {}^G\mathscr{I}\mathscr{F}$ for every $X \in \mathbb{E}$;

(iii) ${}^G\mathscr{I}\mathscr{F}$ is included in $\mathscr{I}\mathscr{F}$.

One can interpret as follows:

$\mathbb{E}$	$\mathscr{F}$	$\mathbb{E}'$	$G : \mathbb{E} \to \mathbb{E}'$
Cp	Π^0_1	Rep	$U : \mathrm{Cp} \to \mathsf{Rep}$

The three conditions (i)-(iii) are certainly fulfilled in this case

The Second One: 2/2

For each morphism $(\cdot \xrightarrow{t} u) \in \mathscr{T}_{\mathrm{Cp}}$, one has the following equivalence:

$$t \in {}^{U}\mathscr{I}\Pi^0_1 \iff \text{range}(t) \text{ is oracle r.e.-closed in } u$$

Conclusion

- We reformurated the proposition concerning with the equivalence of oracle co-r.e. closedness and topological closedness on a categorical setting

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- One can obtain a result which generalize the original proposition in an application of our main theorem
- Further problem:
  Construct the functor $G : \mathbb{E} \to \mathbb{E}'$ depending only on $\mathbb{E}$

Thank you for listening.