Parameterized Uniform Complexity in Numerics: from Smooth to Analytic, from $\mathcal{NP}$-hard to Polytime

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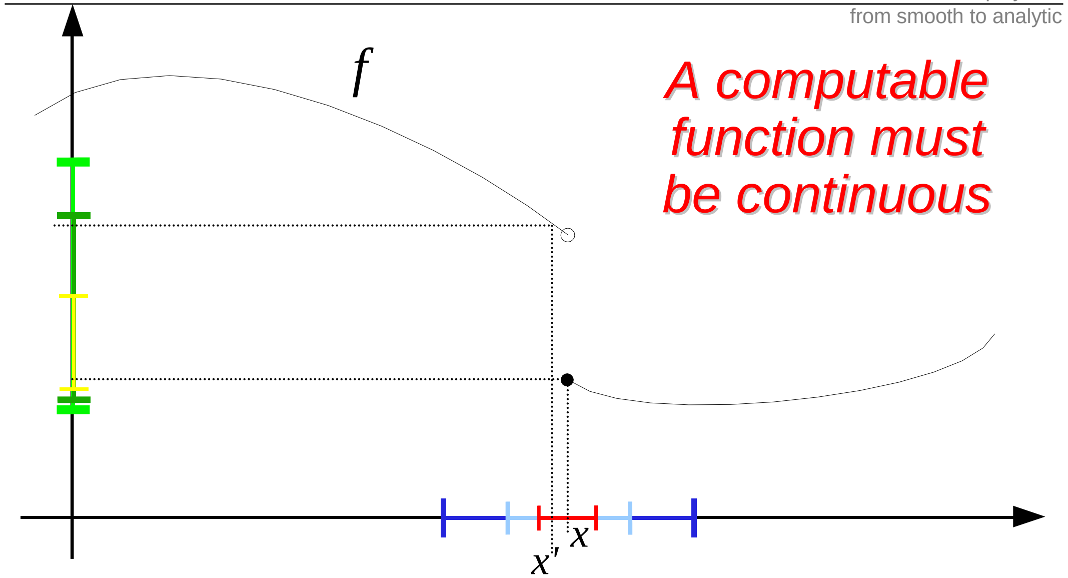
Computable Real Functions



A computable function must be continuous

$x\in\mathbb{R}$ computable $\Leftrightarrow |x-a_n/2^{n+1}|\leq 2^{-n}$ for recursive $(a_n)\subseteq\mathbb{Z}$

Real Function <u>Complexity</u>

Function $f:[0,1]\to\mathbb{R}$ computable $\boxed{\text{in time } t(n)}$ if some TM can, on input of $n\in\mathbb{N}$ and of $\boxed{(a_m)\subseteq\mathbb{Z} \text{ with } |x-a_m/2^{m+1}|<2^{-m}}$ $=:\rho_{dy}\equiv_p\rho_{sd}$ in time $t(n)$ output $b\in\mathbb{Z}$ with $|f(x)-b/2^{n+1}|<2^{-n}$.

iRRAM (GMP/MPFR)

Examples: a) $+, \times, \exp$ polytime on $[0;1]$!

b) $f(x)\equiv\sum_{n\in L} 4^{-n}$ iff $L\subseteq\{0,1\}^*$ polytime-decidable

c) sign, Heaviside not computable

c) $1/\ln(e/x)$ not polytime-computable

Observation i) If f computable $\Rightarrow$ continuous.
ii) If f computable in time $t(n)$, then $t(n+2)$ is a modulus of uniform continuity of f.

$\mathbb{D}_n := \{ k/2^n : k\in\mathbb{Z} \}$, $\mathbb{D} = \bigcup_n \mathbb{D}_n$ dyadic rationals

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3 Effects in Real Complexity

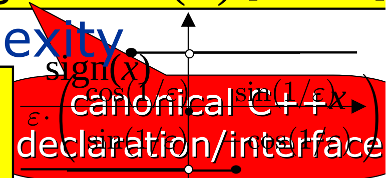
- Consider multivalued 'functions' with
- additional <u>discrete</u> data ('enrichment').

Example c1): exp not computable on entire $\mathbb{R}$,
c2) Evaluation $(f,x)\to f(x)$ is not computable in time depending <u>only</u> on output precision n.

Example b): Given real symmetric $d\times d$ matrix A, find <u>an</u> eigenvector: incomputable; but computable when knowing Card $\sigma(A)$ [Z+B'04]

- parameterized real complexity

Example a): Tests for in-/equality are undecidable



Nonuniform Complexity of Operators

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f:[0;1]→[0;1] polytime computable (⇒ continuous)

- Max: $f \to \mathrm{Max}(f)$: $x \to \max\{\, f(t): t \le x\}$

  Max(f) computable in exponential time; polytime-computable iff $\mathcal{P}=\mathcal{NP}$

- $\int$: $f \to \int f$: $x \to \int_0^x f(t)\,dt$

  $\int f$ computable in exponential time; "#$\mathcal{P}$-complete"

non-uniform

even when restricting to $f \in C^\infty$ **but for analytic f polytime**

Negativistic !!

[Friedman&Ko'80ies]

- dsolve: C[0;1]×[-1;1] ∋ $f \to z$: $\dot z(t)=f(t,z)$, $z(0)=0$.
  - in general no computable solution $z(t)$
  - for $f \in C^1$ "$\mathcal{PSPACE}$-complete"
  - for $f \in C^k$ "$\mathcal{CH}$-hard"

Positivistic !!

Phase transition $C^\omega \to C^\infty$

'Max is $\mathcal{NP}$-hard'

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From $\mathcal{NP}$-hard to polytime from smooth to analytic

$\mathcal{NP} \ni L = \{\, \underline{x} \in \{0,1\}^n \mid \exists \underline{y} \in \{0,1\}^{p(n)}: \langle \underline{x},\underline{y}\rangle \in V \,\}$

1 C^∞ 'pulse' function

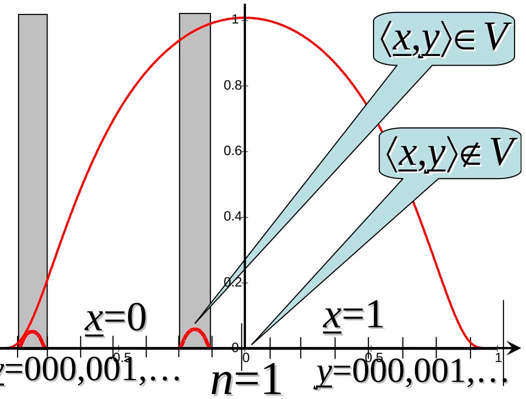
$\varphi(t) = \exp(-t^2/1-t^2)$, $|t|<1$

½ polytime computable

$d^n/dx^n\ \varphi(C\cdot x-z)/C^n \le ?$

¼ *too, uniformly in C,z*

$\underline{x}$= 00 01 10 11 n=2



To every $L \in \mathcal{NP}$ there exists a polytime computable C^∞ function f_L:[0,1]→ℝ s.t.: [0,1]∋$y \to \max f_L|_{[0,y]}$ polytime iff $L \in \mathcal{P}$

Representing Power Series

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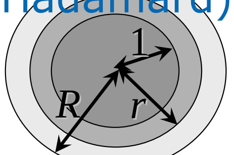
From $\mathcal{NP}$-hard to polytime from smooth to analytic

$\sum_j c_j z^j$

incomputable [ZhWe'01]

- radius of convergence $R=1/\limsup_j |c_j|^{1/j}$
- to $0<r<R$ exist $C \in \mathbb{N}$: $|c_j| \le C/r^j$ (Cauchy-Hadamard)
- $\mathbb{N} \ni K \ge 1/\log(r) = \Theta(1/(r-1))$ binary unary
- tail bound $|\sum_{j \ge N} c_j z^j| \le C\cdot(|z|/r)^N/(1-|z|/r)$



Complexity uniform in $|z| \le 1$: (i.e. $R>1$)
Convergence degrades es as $r \to 1$; quantitatively?

parametrized running time

Theorem 1: Represent series $\sum_j c_j z^j$ with $R>1$ as [a $(\rho_{dy})^\omega$-name of] (c_j) and $K,C \in \mathbb{N}$ as above. The following are uniformly computable in time polyn. in $n+K+\log(C)$: i) eval, ii) sum, iii) product, iv) derivative, v) anti-derivative, vi) Max

Parameterized Real Complexity

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From $\mathcal{NP}$-hard to polytime from smooth to analytic

- Classical complexity theory: worst-case over all inputs of length n as parameter
- parametrized complexity (FPT etc): 2 param.s (n,k)
- Complexity of a single real: n = output precision
- of a real function f: n = output precision in worst-case over all arguments $x \in$ [0;1] compact!
- or **parameterized** – e.g. in $k=\lceil |x| \rceil$ or $k=\lceil \log |x| \rceil$
- TTE: encode x as *in*finite binary sequence, length=∞
- $\underline{x}=(x_j)$ real sequence: access time polynom. in $n+j$

Must 'skip' over 2^n entries to access $f(2^{-n})$

- Real operator/functional Λ: encode input $f \in$ Lip[0;1]
- as values on dense sequence $0,1,½,¼,¾,⅛,⅜,⅝,\ldots=:\mathbb{D}$
- and Lipschitz constant $\ell \in \mathbb{N}$ as discrete data & advice

2^{nd} Order Representations

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Meta-**Def**: representation of G is a surject. $\gamma:\subseteq 2^\omega \to G$

γ-name $z:1^*\to\{0,1\}$ communic. on tape	access time= input length	Γ-name $Z:\{0,1\}^*\to\{0,1\}^*$ communic. via **oracle**

Kawamura&Cook'10 generalize to surject. $\Gamma:\subseteq\omega^\omega\to G$,

$\mathrm{dom}(\Gamma)\subseteq\mathcal{LM} := \{\, Z:\{0,1\}^*\to\{0,1\}^*,\ |Z(\underline{u})|\le|Z(\underline{v})|\ \ \forall|\underline{u}|\le|\underline{v}|\,\}$

$\Rightarrow$ extend from sequential to (realistic) random access

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2^{nd} Order Polyn.s & Time

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'Long' names Z require much time to even read – cmp. evaluation of 'steep' real functions…

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$\Rightarrow$ extend from sequential to (realistic) random access

$K:=|Z|:\mathbb{N}\to\mathbb{N},\ |\underline{u}|\to|Z(\underline{u})|$ well-defined

term over $\mathbb{N},+,\times,n,K()$

- Consider 2nd order representation of $C[0;1]$ s.t. steep functions have long names
- Permit 2nd order polynomial running times $P(n,K)$

$\Rightarrow$ closed under (both kinds of) composition, generalizes (parameterized) 1^{st} order polynom. time

Real Analytic Functions on [0,1]

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Definition: $C^\omega[-1,1] := \{\, f:[-1;1]\to\mathbb{R}$ restriction of complex differentiable $g:U\to\mathbb{C}$, $[0,1]\subseteq U\subseteq\mathbb{C}$ open $\}$

- real sequence $f(\mathbb{D})$
- $L\in\mathbb{N}$ unary: $R_L\subseteq U$
- $G\in\mathbb{N}$ binary $\forall z\in R_L$: $|g(z)|\le G$

2nd order representation

Theorem 2: These are mutually 2^{nd} ord. polytime equivalent

Equivalent: $f\in C^\infty[-1;1]$ and $\exists k\in\mathbb{N}\ \ \forall j$: $\|f^{(j)}\|\le 2^k\cdot k^j\cdot j!$

- real sequence $f(\mathbb{D})$ and $k\in\mathbb{N}$ unary

Equiv.: f finitely many local power series on $[-1;1]$

$\sum_j c_{j,m}(z-x_m)^j$, $m=1\ldots M$ unary binary $C_m, K_m\in\mathbb{N}$: $|c_{j,m}|\le C_m/2^{j/K_m}$

Theorem 3: On $C^\omega[0,1]$, i) eval ii) sum … vi) max are computable within parameterized polyn. time

Overview

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- Complexity of real functions
- *Non*-uniform complexity of real operators:
- $\mathcal{NP}$-hard on C^∞, polytime on analytic ($=C^\omega$)
- *Enrichment* rendering power series computable
- in *parameterized* polynomial time.
- *2^{nd}* order representation rendering computable
- real analytic functions in *2^{nd}* order polytime.
- Gevrey's function hierarchy between C^ω and C^∞
- and 2^{nd} order representations with complexity.

Gevrey's Function Hierarchy

Definition (Maurice Gevrey 1918, studying PDEs)

$g \in \mathcal{G}^\ell_k[-1;1] :\Leftrightarrow \ \forall j: \|g^{(j)}\| \leq 2^k \cdot k^j \cdot j^{j\cdot\ell}$

2nd order repr.

Labhalla&Lombardi&Moutai 2001:

• real sequence $f(\mathbb{D})$ and unary mapping $\mathbb{N}\ni n \to k+n^\ell$

$\Rightarrow \exists B \ \forall n \ \exists p\in\mathbb{D}[X]: \deg(p)<B\cdot n^\ell \ \ \|g-p\|\leq 2^{-n} \ \Rightarrow g\in \mathcal{G}^{2\ell-1}[-1;1]$

• sequence $p_n \in \mathbb{D}[X]$ with $\deg(p_n)<B\cdot n^\ell \ \ \|g-p_n\|\leq 2^{-n}$

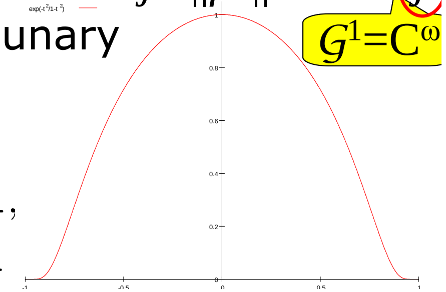
Equivalent: $f\in C^\infty[-1;1]$ and $\exists k\in\mathbb{N} \ \ \forall j: \ \|f^{(j)}\| \leq 2^k\cdot k^j \cdot j!$

$\mathcal{G}^1 = C^\omega$

• real sequence $f(\mathbb{D})$ and $k\in\mathbb{N}$ unary

Example: The following g is not analytic but in $\mathcal{G}^3[-1;1]$

$$g(x) := \exp\left(\frac{x^2}{x^2-1}\right) \text{ for } |x|\leq 1,$$
$$g(x) := 0 \qquad \text{for } |x|\geq 1$$



Uniform Complexity on Gevrey's Hierarchy

Definition (Maurice Gevrey 1918, studying PDEs)

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Theorem 4 (our main result):

a) Both 2nd order representations of $\bigcup_{k,\ell} \mathcal{G}^\ell_k$ are 2nd order polynomial-time equivalent and

b) render i) eval, ii) sum, … iv) d/dx, v) $\int$, vi) max computable within time polynomial in $(k+n)^{\mathrm{poly}(\ell)}$

c) Given $f(\mathbb{D})$, max on $\mathcal{G}^\ell_1$ requires time $\Omega(n^\ell)$.

Conclusion and Perspectives

• Max and $\int$ are nonuniformly $\mathcal{NP}$-hard on $C^\infty[-1;1]$

• but nonunif. polytime on $C^\omega[-1;1]$, i.e. analytic f.

Today: <u>uniform</u> computability and <u>parameterized</u> complexity of operators on Gevrey's hierarchy $\mathcal{G}^\ell$ climbing from C^ω to C^∞ with <u>optimal</u> runtime $n^{\mathrm{poly}(\ell)}$

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1st order

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• Actually implement and evaluate these algorithms (`iRRAM`)

• Quantitatively refine the upper complexity bounds

• Multivariate case?