Reductions strong and uniform: an update on SRT^2_2 and COH

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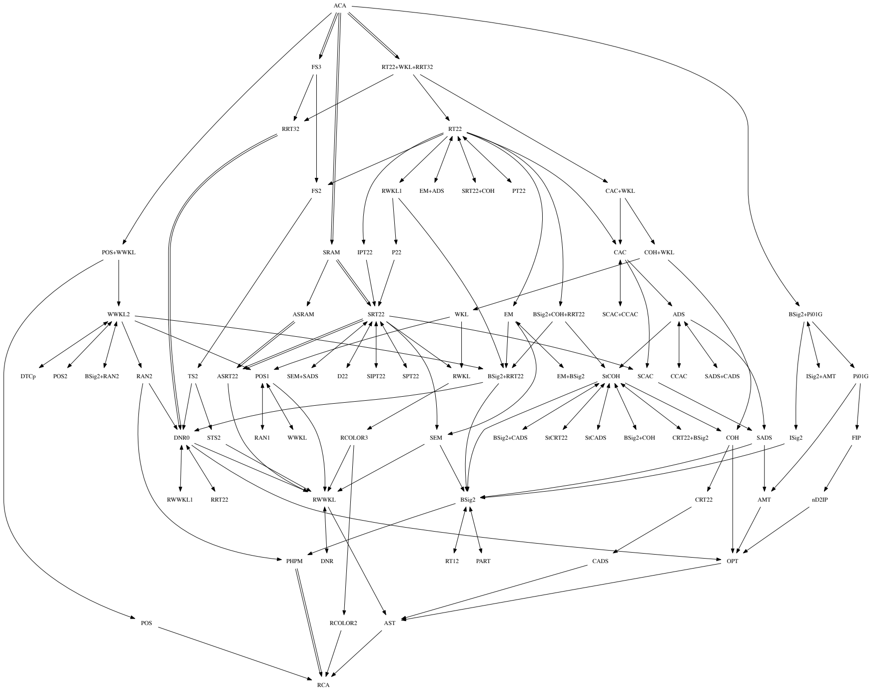
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Basic question.

We wish to understand the logical content of Ramsey's theorem.

RT^n_k. Every $f : [\mathbb{N}]^2 \to k$ has an infinite homogeneous set.

The cases $n = 1$ and $n \geq 3$ are well-understood. Not so for $n = 2$.

Recall:

A coloring $f : [\mathbb{N}]^2 \to k$ is stable if $\lim_y f(x, y)$ exists for every x.

A set S is cohesive for $\langle A_i : i \in \mathbb{N} \rangle$ if $S \cap A_i$ or $S \cap \overline{A_i}$ is finite for every i.

SRT^2_k. Every stable $f : [\mathbb{N}]^2 \to k$ has an infinite homogeneous set.

COH. Every $\langle A_i : i \in \mathbb{N} \rangle$ has an infinite cohesive set.

Cholak, Jockusch, and Slaman; Mileti. $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_k \leftrightarrow \mathsf{SRT}^2_k + \mathsf{COH}$.

Basic question.

The decomposition has been used to obtain many results about RT^n_k.

Seetapun. Every computable $f : [\mathbb{N}]^2 \to k$ has an infinite homogeneous set that does not compute a given non-computable set.

Cholak, Jockusch, and Slaman. Every computable $f : [\mathbb{N}]^2 \to k$ has a low$_2$ infinite homogeneous set.

Dzhafarov and Jockusch. Every computable $f : [\mathbb{N}]^2 \to k$ has two low$_2$ infinite homogeneous sets whose Turing degrees form a minimal pair.

Liu. Every computable $f : [\mathbb{N}]^2 \to k$ has an infinite homogeneous set that is not of PA Turing degree.

Hirschfeldt and Shore. Similar decompositions for ADS and CAC.

Basic question.

A long-standing open question asked whether this decomposition is proper.

Chong, Slaman, Yang. COH is not implied by SRT^2_k over RCA_0.

This shows that COH and SRT^2_k have different proof-theoretic content. But the separation is via a very non-standard model. As such, it leaves open the question of the computability-theoretic relationship of these principles.

Question. Is every ω-model of SRT^2_k a model of COH?

The result of Chong, Slaman, and Yang suggests that if the answer is yes, it should be via some kind of complicated construction. For instance, their model satisfies $\mathsf{B}\Sigma^0_2$, which usually suffices to formalize finite injury arguments.

But in principle, the shape of the proof could be simple.

Computable and Weihrauch reductions.

Let P and Q be Π^1_2 principles.

Q $\leq_c$ P if every instance A of Q computes an instance B of P, such that if S is any solution to B then $A \oplus S$ computes a solution T to A.

Q $\leq_{sc}$ P if every instance A of Q computes an instance B of P, such that if S is any solution to B then S computes a solution T to A.

If the reduction from A to B uniform, and the reduction from S, or from $A \oplus S$, to T is uniform, then $\leq_c$ becomes $\leq_W$, and $\leq_{sc}$ becomes $\leq_{sW}$.

Virtually all implications in RCA_0 between Π^1_2 principles are formalizations of computable reductions. In fact, almost all are Weihrauch, and many are strong Weihrauch. Often, the backwards reduction is the identity.

Computable and Weihrauch reductions.

	strong	weak
uniform	$\leq_{sW}$	$\leq_{W}$
non-uniform	$\leq_{sc}$	$\leq_{c}$

Computable and Weihrauch reductions.

Note: strong reductions need not be simple as arguments.

Clever combinatorial arguments:

Cholak, Jockusch, and Slaman. $\mathsf{COH} \leq_{sW} \mathsf{RT}^2_2$.

Hirschfeldt and Shore. $\mathsf{COH} \leq_{sW} \mathsf{ADS}$.

Dzhafarov and Hirst. $\mathsf{RT}^2_2 \equiv_c \mathsf{PT}^2_2$.

Heavy computability-theoretic arguments:

Hirschfeldt, Shore, and Slaman. $\mathsf{AMT} \leq_{sW} \mathsf{SADS}$.

Dzhafarov and Mummert. $\mathsf{OPT} \leq_{sW} \mathsf{FIP}$.

In all these cases, the way instances of the one problem are computed (coded) into instances of the other is not straightforward.

Computable and Weihrauch reductions.

Strong reductions give a finer analysis of known implications.

Jockusch. For all k, $\mathsf{DNR}_k \not\leq_W \mathsf{DNR}_{k+1}$.

Dorais, Dzhafarov, Hirst, Mileti, and Shafer.

For $j < k$, $\mathsf{RT}^n_k \not\leq_{sW} \mathsf{RT}^n_j$.

$\mathsf{RT}^1_2 \not\leq_W \mathsf{RRT}^2_k$.

Dorais, Dzhafarov, Hirst, Mileti, and Shafer; Brattka, Gherardi, and Hölzl.

For $p < q \leq 1$, p-$\mathsf{WWKL}_0 \not\leq_W q$-$\mathsf{WWKL}_0$.

Lack of strong reductions lends credence to open non-implications.

Dorais, Dzhafarov, Hirst, Mileti, and Shafer; Hirschfeldt and Jockusch.

For $j < k$, $\mathsf{TS}^n_j \not\leq_W \mathsf{TS}^n_k$.

SRT^2_k and COH.

We want to analyze SRT^2_k and COH in this setting.

Question. Does COH reduce to SRT^2_k in any of the above ways?

One reason why this is difficult—and more generally, why contrasting SRT^2_k and COH is difficult—is that the these principles are actually quite similar. Namely, they can both be expressed as variants of RT^1_k:

SRT^2_k is RT^1_k for Δ^0_2-definable colorings.

COH is SeqRT^1_k with homogeneous sets allowed to make finitely many errors.

These definitions are Weihrauch equivalent to the originals, and this can be formalized to give equivalences in RCA_0. (Though for SRT^2_k, it takes quite a lot of work to make this formalization go through in $\mathsf{I}\Sigma^0_1$ (Chong, Lempp, Yang).)

SRT^2_k and COH.

The easiest way to show that a strong reduction does not hold between two principles is to exhibit a degree-theoretic difference between them.

$\mathsf{RT}^2_k \not\leq_c \mathsf{SRT}^2_k$.

$\mathsf{RT}^2_k \not\leq_c \mathsf{COH}$.

Jockusch. There is a computable instance of RT^2_k with no Δ^0_2 solution. Every computable instance of SRT^2_k or COH has a Δ^0_2 solution.

$\mathsf{SRT}^2_k \not\leq_c \mathsf{COH}$.

Hirschfeldt, Jockusch, Kjos-Hanssen, Lemp, and Slaman. There is a computable instance of SRT^2_k every solution to which has DNR degree. But this is not true of COH.

Every known degree fact about COH holds of some instance of SRT^2_k.

SRT^2_k and COH.

COH would be strongly computably reducible to SRT^2_k if given $A = \langle A_i : i \in \mathbb{N}\rangle$, there were a k and a partition $\langle B_0, \ldots, B_{k-1}\rangle$ of $\mathbb{N}$ that would be $\mathbf{\Delta}^0_2$ in A, such that any infinite subset of any B_i would compute a cohesive set for A.

Key example.

Consider a finite family, $A = \langle A_0, \ldots, A_{n-1}\rangle$.

Then the above holds with $k = 2^n$, and the partition consisting of the 2^n many Boolean combinations of the A_i under intersection and complementation.

Here there is a considerably stronger reduction. The partition is computable, not merely $\mathbf{\Delta}^0_2$, in A, and any infinite subset of any B_i is itself a cohesive set for A, rather than just computing one.

Cohesive avoidance.

What happens if we look instead at smaller partitions $\langle B_0, \ldots, B_{k-1} \rangle$, $k < 2^n$? Certainly it will not be the case that any infinite subset of any B_i just is cohesive. We might also expect to have to increase the complexity of the partition.

Dzhafarov. Fix n and $k < 2^n$. There is a family $A = \langle A_0, \ldots, A_{n-1} \rangle$ such that for any partition $\langle B_0, \ldots, B_{k-1} \rangle$ of $\mathbb{N}$, (hyper)arithmetical in A, there is an infinite subset of one of the B_i that computes no cohesive set for A.

In particular, this is true for partitions that are $\mathbf{\Delta}^0_2$ in A.

Corollary. For all k, COH $\nleq_{sc}$ SRT^2_k.

We can use this as a module to build more complicated instances of COH.

Corollary. COH $\nleq_{sc}$ SRT^2.

Cohesive avoidance.

Idea of proof. Take $n = k = 2$. The proof is an iterated forcing argument.

We build $A = \langle A_0, A_1 \rangle$ by Cohen forcing.

For each arithmetical functional Φ such that Φ^A is a stable coloring, we build a pair of homogeneous sets, H_0 and H_1, by Mathias forcing.

For each pair of Turing functionals Γ_0, Γ_1, we make $\Gamma_0^{H_0}$ or $\Gamma_1^{H_1}$ not cohesive.

Fix $\langle u_0, u_1 \rangle \in \{0, 1\}^2$. We repeatedly ensure there is a large x in either $\Gamma_0^{H_0}$ or $\Gamma_1^{H_1}$ with $A_0(x) = u_0$ and $A_1(x) = u_1$.

By a counting argument, one of $\Gamma_0^{H_0}$ and $\Gamma_1^{H_1}$ intersects A_0 and $\overline{A_0}$ infinitely often, or A_1 and $\overline{A_1}$ infinitely often.

Iterability and uniformity.

The construction of the H_i is essentially independent of A. We extend H_i to find a new computation witnessing that $x \in \Gamma_i^{H_i}$, and then define $A_0(x)$ and $A_1(x)$ as needed.

(We do need A to force facts about H_i, but we can delay these.)

This makes the forcing very robust, and insensitive to the addition of new sets, so we can easily iterate it.

But since we are dealing with strong reductions, iterating does not produce a sequence of reals closed under join.

To move to weak reductions, we need to look instead at computations witnessing that $x \in \Gamma_i^{A \oplus H_i}$. In other words, the bits of A_0 and A_1 now need to be built along with H_i. This is much harder.

Iterability and uniformity.

We can solve this problem by adding a little uniformity:

Fix Φ. Build an infinite family $\langle A_i : i \in \mathbb{N} \rangle$, and look at the stable coloring Φ^A.

For each pair of functionals Γ_0, Γ_1 and potential homogeneous sets H_0, H_1, designate a pair of columns to play our strategy from before on.

Given $\langle u_0, u_1 \rangle \in \{0,1\}^2$, "lock" these columns by extending H_0 only by u_0, and H_1 only by u_1, until we find a computation showing $x \in \Gamma_i^{A \oplus H_i}$.

The uniformity ensures that the H can be made homogeneous.

Dzhafarov. For each Φ, there is a family $\langle A_i : i \in \mathbb{N} \rangle$ such that if Φ^A is a stable coloring then it has an infinite homogeneous set H such that $A \oplus H$ computes no cohesive set for A.

Corollary. $\mathrm{COH} \nleq_W \mathrm{SRT}^2$.

Iterability and uniformity.

In connection with their work on the tournament principle, Lerman, Solomon, and Towsner produced a direct forcing argument that $\mathsf{RT}^2_k \nleq_c \mathsf{SRT}^2_k$. Unfortunately, their proof does not iterate.

The problem in their argument is the same as with extending the uniform result above: there is too much feedback between the layers of the forcing.

The following question thus remains open:

Open question. Is it the case that $\mathsf{COH} \nleq_c \mathsf{SRT}^2_k$?

An iterable and relativiziable affirmative answer should give an ω-model of $\mathsf{COH} + \neg\mathsf{SRT}^2$.

Thank you!