Well-scattered partial orders and Erdös-Rado

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Well-partial orders vs well-scattered partial orders

P is a well-partial order if for every function $f : \mathbb{N} \to P$ there exist $x < y$ such that $f(x) \leq_P f(y)$.

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Theorem (folklore)

Let P be a partial order. Then the following are equivalent:

1. *P is well-founded and has no infinite antichains* wpo(ant)*;*
2. *every linear extension of P is well-founded* wpo(ext)*;*
3. *P is a well-partial order* wpo*;*
4. *for every function $f : \mathbb{N} \to P$ there exist an infinite set $A \subseteq \mathbb{N}$ such that $x < y$ implies $f(x) \leq_P f(y)$ for all $x, y \in A$* wpo(set)*.*

Well-partial orders vs well-scattered partial orders

We say that P is a well-scattered partial order if if for every function $f\colon \mathbb{Q} \to P$ there exist $x <_{\mathbb{Q}} y$ such that $f(x) \leq_P f(y)$.

Well-partial orders vs well-scattered partial orders

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Theorem (Bonnet, Pouzet 69)

Let P be a partial order. The following are equivalent:

1. *P is scattered and has no infinite antichains* wspo(ant)*;*
2. *every linear extension of P is scattered* wspo(ext)*;*
3. *P is a well-scattered partial order* wspo*;*
4. *for every function $f : \mathbb{Q} \to P$ there exists an infinite set $A \subseteq \mathbb{Q}$ such that $x <_{\mathbb{Q}} y$ implies $f(x) \leq_P f(y)$ for all $x, y \in A$* wspo(set)*.*

Reverse mathematics

For every pair of equivalent conditions Φ and Ψ, consider the statement:

$$\Phi \to \Psi: \quad (\forall P)(\Phi(P) \implies \Psi(P)).$$

For instance, wspo $\to$ wspo(ext) denotes the statement “if P is a well-scattered partial order, then every linear extension of P is scattered”.

Known results

wpo(set)

$\mathsf{RCA}_0 \vdash$

$\downarrow$

wpo

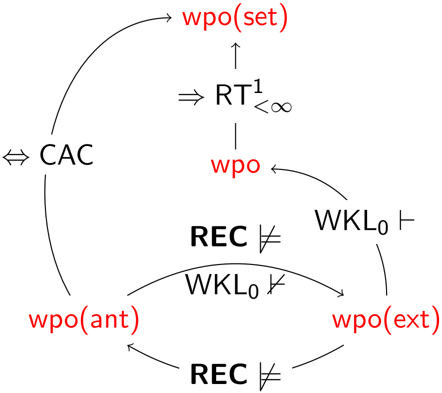
$\mathsf{RCA}_0 \vdash$ $\qquad$ $\mathsf{RCA}_0 \vdash$

$\swarrow$ $\qquad$ $\searrow$

wpo(ant) $\qquad$ wpo(ext)

Cholak, Marcone and Solomon 2004

Known results



Cholak, Marcone and Solomon 2004

New results

$$
\begin{tikzcd}
 & \text{wpo} & \\
\text{wpo(ant)} & & \text{wpo(ext)} \arrow[ul, "\mathsf{WKL}_0 \Rightarrow"'] \arrow[ll, "\mathbf{REC} \not\models"']
\end{tikzcd}
$$

before...

New results

$$
\begin{tikzcd}
& \color{red}{\mathrm{wpo}} & \\
\color{red}{\mathrm{wpo(ant)}} & & \color{red}{\mathrm{wpo(ext)}} \arrow[ul, "\mathsf{WKL}_0 \Rightarrow"'] \arrow[ll, "\mathbf{REC} \not\models"', "\mathsf{WWKL}_0 \nvdash"]
\end{tikzcd}
$$

...after

Theorem (Frittaion, 2014)

There exists a computable partial order P with an infinite computable antichain such that the set $\mathcal{A}$ of reals computing a linear extension L of P and an infinite descending sequence in L is null.

Corollary

WWKL_0 *does not prove* $\mathsf{wpo(ext)} \rightarrow \mathsf{wpo(ant)}$.

Build an ω-model of WWKL_0 by taking a Martin-Löf random real not in $\mathcal{A}$.

Semitransitive colorings on natural numbers

A coloring $c\colon [\mathbb{N}]^2 \to n$ is transitive on $i < n$ if $c(x, y) = c(y, z) = i$ implies $c(x, z) = i$ for all $x < y < z$.
We say that c is semitransitive if it is transitive on every $i > 0$.

For all $n \geq 2$, let:

st-RT^2_n: every semitransitive coloring $c\colon [\mathbb{N}]^2 \to n$ has an infinite homogeneous set.

Theorem (Hirschfeldt, Shore 2007)

For all $n \geq 2$, RCA_0 *proves* CAC $\Leftrightarrow$ st-RT^2_n.

New results

$$
\begin{tikzcd}
 & \color{red}{\mathrm{wpo(set)}} & \\
 & \color{red}{\mathrm{wpo}} \arrow[u, "\mathrm{CAC} \Rightarrow", "\Rightarrow \mathrm{RT}^1_{<\infty}"'] & \\
\color{red}{\mathrm{wpo(ant)}} \arrow[ur, bend left, "\mathrm{CAC} \Rightarrow"] \arrow[rr, "\mathrm{CAC} \Rightarrow"] & & \color{red}{\mathrm{wpo(ext)}}
\end{tikzcd}
$$

before...

New results

$$
\begin{tikzcd}
 & \text{wpo(set)} & \\
 & \text{wpo} \arrow[u, "\Leftrightarrow \mathsf{CAC}"] & \\
\text{wpo(ant)} \arrow[ur, bend left, "\Leftrightarrow \mathsf{CAC}"] \arrow[rr, "\Leftrightarrow \mathsf{CAC}"] & & \text{wpo(ext)}
\end{tikzcd}
$$

...after

Erdös and Rado: a partition relation

CAC is a consequence of RT^2_2.

The analogue of RT^2_2 in the case of well-scattered partial orders is the following:

Theorem (Erdös, Rado 52)

$\mathbb{Q} \to (\aleph_0, \mathbb{Q})^2$. *That is, for every coloring* $c \colon [\mathbb{Q}]^2 \to 2$ *there exists either an infinite* 0*-homogeneous set or a dense* 1*-homogeneous set.*

We denote it by ER^2_2 .

Results for well-scattered partial orders

$$
\begin{tikzcd}
& \mathsf{wspo(set)} \arrow[d, "\mathsf{RCA}_0 \vdash"] & \\
& \mathsf{wspo} \arrow[dl, "\mathsf{RCA}_0 \vdash"'] \arrow[dr, "\mathsf{RCA}_0 \vdash"] & \\
\mathsf{wspo(ant)} & & \mathsf{wspo(ext)}
\end{tikzcd}
$$

Results for well-scattered partial orders

$$
\begin{tikzcd}
 & \textcolor{red}{\mathrm{wspo(set)}} & \\
 & \textcolor{red}{\mathrm{wspo}} \arrow[u, "\Rightarrow \mathrm{CAC}"] & \\
 & & \\
\textcolor{red}{\mathrm{wspo(ant)}} \arrow[uuur, bend left, "\boxed{\mathrm{ER}^2_2 \Rightarrow}"] \arrow[rr, bend left, "\mathbf{REC} \not\models", "\mathrm{WKL}_0 \nvdash"'] & & \textcolor{red}{\mathrm{wspo(ext)}} \arrow[uul, bend right, "\mathrm{WKL}_0 \vdash"'] \arrow[ll, bend left, "\mathbf{REC} \not\models"', "\mathrm{WWKL}_0 \nvdash"]
\end{tikzcd}
$$

Semitransitive colorings on rationals

Let L be a linear order. A coloring $c\colon [L]^2 \to n$ is transitive on $i < n$ if $c(x,y) = c(y,z) = i$ implies $c(x,z) = i$ for all $x <_L y <_L z$.
c is semitransitive if it is transitive on every $i > 0$.

For all $n \geq 1$, we consider the statement:
st-ER^2_{n+1}: every semitransitive coloring $c\colon [\mathbb{Q}]^2 \to n+1$ has either an infinite i-homogeneous set for some $i < n$ or a dense n-homogeneous set.

Lemma

RCA_0 *proves:*

1. $(\forall n \geq 1)(\text{st-ER}^2_{n+2} \Rightarrow \text{st-ER}^2_{n+1})$;
2. $\text{st-ER}^2_3 \Leftrightarrow \text{st-ER}^2_2 \wedge \text{st-RT}^2_2$;
3. $(\forall n \geq 2)(\text{st-ER}^2_{n+1} \Rightarrow \text{st-ER}^2_{n+2})$;
4. $\text{st-ER}^2_2 \Rightarrow \text{RT}^1_{<\infty}$.

Let us show (2). We prove $\text{st-ER}^2_3 \Rightarrow \text{st-RT}^2_2$. Let $c\colon [\mathbb{N}]^2 \to 2$ be a semitransitive coloring and define $d\colon [\mathbb{Q}]^2 \to 3$ by letting for all $x <_{\mathbb{Q}} y$

$$d(x,y) := \begin{cases} 0 & \text{if } c(x,y) = 0, \\ 1 & \text{if } c(x,y) = 1 \wedge x < y, \\ 2 & \text{if } c(x,y) = 1 \wedge x > y. \end{cases}$$

It is straightforward to see that d is semitransitive. Now, any homogeneous set for d, infinite or dense, is an infinite homogeneous set for c.

For the other direction, let $c\colon [\mathbb{Q}]^2 \to 3$ be semitransitive. We thus define $d\colon [\mathbb{Q}]^2 \to 2$ by setting for all $x <_{\mathbb{Q}} y$

$$d(x,y) := \begin{cases} 0 & \text{if } c(x,y) < 2, \\ 1 & \text{if } c(x,y) = 2. \end{cases}$$

d is semitransitive and so we apply st-ER^2_2. If D is a dense 1-homogeneous set for d, then D is a dense 2-homogeneous set for c and we are done.

Suppose now $A \subseteq \mathbb{Q}$ is an infinite 0-homogeneous set for d. Therefore, $x <_{\mathbb{Q}} y$ implies $c(x, y) < 1$ for all $x, y \in A$. Since c is semitransitive, we define a partial order on A by letting $x <_A y$ if and only if $x <_{\mathbb{Q}} y$ and $c(x, y) = 1$.

By CAC, which is equivalent to st-RT^2_2, A contains either an infinite antichain, which is an infinite 0-homogeneous set for c, or an infinite chain, which is an infinite 1-homogeneous set for c.

Results for well-scattered partial orders

$$\begin{tikzcd}
 & \mathsf{wspo(set)} & \\
 & \mathsf{wspo} & \\
\mathsf{wspo(ant)} \arrow[ur, bend left, "\Leftrightarrow \text{st-ER}^2_3"] \arrow[rr, "\Leftrightarrow \text{st-ER}^2_2"] & & \mathsf{wspo(ext)}
\end{tikzcd}$$

Example

Lemma

Over RCA_0, *the following are equivalent:*

(1) st-ER^2_2;

(2) wspo(ant) $\to$ wspo(ext).

(1) $\Rightarrow$ (2) Assume st-ER^2_2 and let P be a partial order. We prove the contrapositive of wspo(ant) $\to$ wspo(ext). So let L be a nonscattered linear extension of P and $f\colon \mathbb{Q} \to L$ be an embedding. Let us define a semitransitive coloring $c\colon [\mathbb{Q}]^2 \to 2$ by letting for all $x <_{\mathbb{Q}} y$

$$c(x,y) := \begin{cases} 0 & \text{if } f(x) \perp_P f(y), \\ 1 & \text{if } f(x) <_P f(y). \end{cases}$$

If $A \subseteq \mathbb{Q}$ is an infinite 0-homogeneous set, then $\mathrm{ran}(f)$ is an infinite antichain of P. Provably in RCA_0, any $\boldsymbol{\Sigma}^0_1$ infinite set contains a $\boldsymbol{\Delta}^0_1$ infinite subset and hence $\mathrm{ran}(f)$ contains an infinite antichain.

Suppose we have a dense 1-homogeneous set D. Then the restriction of f to D is embedding of a dense linear order into P showing that P is not scattered.

(2) $\Rightarrow$ (1) Let $c \colon [\mathbb{Q}]^2 \to 2$ be semitransitive. By definition c is transitive on 1 and hence we can define a partial order P by letting $x \leq_P y$ if and only if $x = y$ or $x <_{\mathbb{Q}} y$ and $c(x, y) = 1$.

Consequently $x \leq_P y$ implies $x \leq_{\mathbb{Q}} y$ and so $\mathbb{Q}$ is a linear extension of P showing that P does not satisfy wspo(ext). Therefore P does not satisfy wspo(ant). An infinite antichain of P is an infinite 0-homogeneous set. On the other hand, a dense subchain of P is a dense 1-homogeneous set.

State of knowledge and open questions

$$
\begin{tikzcd}
 & \mathsf{ER}^2_2 \arrow[dl] \arrow[dr] & & \\
\mathsf{RT}^2_2 \arrow[dr] & & \text{st-}\mathsf{ER}^2_3 \arrow[dl] \arrow[dr] & \\
 & \text{st-}\mathsf{RT}^2_2 \arrow[dr] & & \text{st-}\mathsf{ER}^2_2 \arrow[dl] \\
 & & \mathsf{RT}^1_{<\infty} &
\end{tikzcd}
$$

State of knowledge and open questions

$$
\begin{tikzcd}[column sep=small]
 & \text{st-ER}^2_3 + \mathsf{WKL}_0 \arrow[dl] \arrow[d] \arrow[dr] & \\
\begin{array}{c}\text{wspo(ant)} \to \text{wspo(set)} \\ = \\ \text{st-ER}^2_3\end{array} \arrow[d] \arrow[dr] & \text{wspo(ext)} \to \text{wspo(set)} \arrow[dl] \arrow[dr] & \mathsf{WKL}_0 \arrow[d] \\
\text{wspo} \to \text{wspo(set)} \arrow[d] & \text{wspo(ant)} \to \text{wspo} \arrow[d] & \text{wspo(ext)} \to \text{wspo} \arrow[d] \\
\text{st-RT}^2_2 & \begin{array}{c}\text{st-ER}^2_2 \\ = \\ \text{wspo(ant)} \to \text{wspo(ext)}\end{array} & \text{wspo(ext)} \to \text{wspo(ant)}
\end{tikzcd}
$$

Thanks for your attention