Classical provability of uniform versions and intuitionistic provability

Makoto Fujiwara
(joint work with Ulrich Kohlenbach)

Mathematical Institute, Tohoku University

CTFM2014
19, February, 2014

Introduction

Many mathematical statements have Π_2 form:

$$\forall X\,(A(X) \to \exists Y\, B(X,Y))\,.$$

Intermediate Value Theorem.

For any continuous function $f : [0,1] \to \mathbb{R}$ s.t. $f(0) < 0 < f(1)$, then there exists a point $m \in [0,1]$ s.t. $f(m) = 0$.

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Sequential versions

- Many Π^1_2 statements are provable in RCA(RCA$_0$+full induction).
- In some of their proofs, however, the construction of the solution Y from given X is not uniform.
- To reveal the non-uniformity, the following sequential version has been investigated.

$$\forall \langle X_n \rangle_{n\in\mathbb{N}} \left(\forall n A(X_n) \to \exists \langle Y_n \rangle_{n\in\mathbb{N}} \forall n B(X_n, Y_n) \right).$$

	Pointwise	Sequential
JD (The existence of Jordan decomposition for real square matrices)	RCA	ACA
IPP (Infinite pigeonhole principle)	RCA	ACA
IVT (Intermediate value theorem)	RCA	WKL
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Uniform Versions

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$$\exists\Phi\forall X(A(X)\to B(X,\Phi(X))).$$

(Note that uniform version implies sequential version.)

- However, for a Π^1_2 sentence, its uniform version is not naturally represented in the language of second-order arithmetic.
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- Hilbert-type system E-HA$^\omega$ (resp. E-PA$^\omega$) is the finite type extension of HA (resp. PA).
- E-PA$^\omega$ $:=$ E-HA$^\omega$ $+ \mathbf{LEM}(\mathrm{A} \vee \neg \mathrm{A})$.
- RCA$^\omega$ $:=$ E-PA$^\omega$ $+ \mathrm{QF\text{-}AC}^{1,0}$.

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Strength of Uniform Versions

- $\mathsf{RCA}^\omega \vdash \mathrm{UWKL} \leftrightarrow \mathrm{UACA}$. (Kohlenbach 2001)
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	Pointwise	Sequential	Uniform (over RCA^ω)
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IPP	RCA	ACA	
IVT	RCA	WKL	
TET	RCA	RCA	RCA^ω

- UWKL is the uniform version of WKL.
- UACA: $\exists E^2 \forall f^1 \left(E(f) = 0 \leftrightarrow \exists x^0 (f(x) = 0)\right).$

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RCA^ω is too strong as base system for investigating uniform versions!

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The Systems with Weak Extensionality

- Our systems have only $=_0$ as predicate symbol and $s^\rho =_\rho t^\rho$ is the abbreviation for

$$\forall v_1^{\rho_1}, \ldots, v_k^{\rho_k} \, (s(v_1 \ldots v_k) =_0 t(v_1 \ldots v_k))$$

  where $\rho = \rho_1 \to \ldots \to \rho_k \to 0$.
- E-PA$^\omega$ have the **extensionality** axiom (E):

$$\forall z^{\rho \to \tau}, x^\rho, y^\rho (x =_\rho y \to z(x) =_\tau z(y)).$$

- WE-PA$^\omega$ (resp. WE-HA$^\omega$) is the subsystem of E-PA$^\omega$ (resp. E-HA$^\omega$) where (E) is replaced by the **weak extensionality rule**:

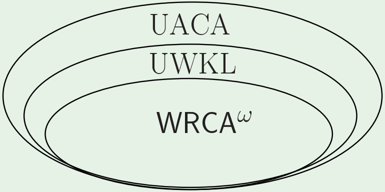
$$\frac{A_{qf} \to s =_\rho t}{A_{qf} \to r^\tau[s/x^\rho] =_\tau r[t/x^\rho]}.$$

- WRCA$^\omega$:= WE-PA$^\omega$ + $\mathrm{QF\text{-}AC}^{1,0}$.
- WRCA$^\omega$ is a conservative extension of RCA.

Strength of Uniform Versions over WRCA$^\omega$

- By comparing the provably recursive functions, we have

$$\mathsf{WRCA}^\omega + \mathrm{UWKL} \nvdash \mathrm{UACA}.$$



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Strength of Uniform Versions over WRCA^ω

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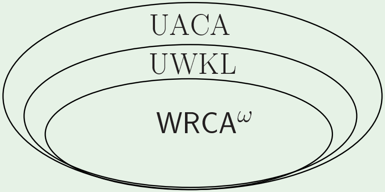

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TET	RCA	RCA	WRCA^ω

Observation from Classical Reverse Mathematics

For the statements non-uniformly provable in RCA, the shift of the strength by uniformization seems to be caused from the use of $\mathbf{LEM} : \mathrm{A} \vee \neg \mathrm{A}$ for undecidable A.

The following result expresses the informal idea that if a Π_2 statement is provable without the use of $\mathbf{LEM}$, then it has a uniform proof.

Theorem. (Hirst-Mummert 2011)

For a Π_2 sentence $\mathrm{S} := \forall x^\rho\,(A(x) \to \exists y^\tau B(x,y))$ where A is purely universal and B has the suitable syntactical form, if

$$\text{WE-HA}^\omega + \mathrm{AC}^\omega + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega \vdash \mathrm{S}, \text{ then}$$

$$\text{WRCA}^\omega \vdash \mathrm{Uni}(\mathrm{S}).$$

- $\mathrm{IP}^{\rho,\tau}_\forall$: $(\forall z^\rho A_{qf} \to \exists x^\tau B(x)) \to \exists x^\tau(\forall z^\rho A_{qf} \to B(x^\rho))$.
- M^ρ : $\neg\neg\exists x^\rho A_0(x) \to \exists x^\rho A_0(x)$.

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Corollary.

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$$\mathsf{RCA} \vdash \mathrm{Seq}(\mathrm{S}).$$

Application.

IVT, IPP, JD are not provable in $\text{E-HA}^\omega + \mathrm{AC}^\omega + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega$.

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Motivating Results

Hierarchy of LEM over HA (Akama et al., 2004)

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 & \Sigma^0_1\text{-LEM} \arrow[dl] \arrow[dr] & \\
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- $\mathrm{M}^0 : \quad \neg\neg\exists x^0 A_{qf} \to \exists x^0 A_{qf}$
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Some equivalences over intuitionistic systems (like WE-HA$^\omega$) have been established.

Proposition. (Ishihara, 2005)

1. $\mathrm{ACA} \leftrightarrow \Sigma^0_1\text{-LEM} + \Pi^0_1\text{-AC}^{0,0}$.
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Question.

Can we extract stronger unprovability for the statement whose sequential version implies ACA rather than only WKL?

Theorem. (Kohlenbach-F.)

For a Π_2 sentence S of the previous syntactical form, if

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Application. (Note that $\text{WRCA}^\omega + \mathrm{UWKL} \nvdash \mathrm{ACA}$.)

IPP, JD are not provable in
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$\text{WE-HA}^\omega + \mathrm{AC}^\omega + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \mathrm{UWKL} + \mathrm{KL}$.

- However, we can extract further stronger unprovability if each uniform version implies UACA over WRCA$^{\omega}$.
- That is the merit to investigate uniform versions rather than sequential versions!

Main Theorem. (Kohlenbach-F.)

For a Π_2 sentence S of the previous syntactical form, if

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- BI^ω is the bar induction scheme in all finite type.

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Remark.

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Tools for the proof of main theorem.

- $\text{WE-HA}^\omega + \mathrm{AC}^\omega + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \mathrm{BR}^\omega$(bar recursion) $\vdash \mathrm{BI}^\omega$ (Howard 1968).
- Negative translation.
- The Dialectica interpretation without extracting terms.
- A non-standard principle F^- related to the fan principle.
- The model $\mathcal{M}^\omega$ of all strongly majorizable functionals.

Corollary. (due to Luckhard's technique)

For a Π^1_2 sentence S of the previous syntactical form, if

$$\textsf{WRCA}^\omega + \mathrm{Uni}(\mathrm{S}) \vdash \mathrm{UACA}, \text{ then}$$

$$\textsf{E-HA}^\omega + \mathrm{AC}^{\omega *} + \mathrm{IP}^{1,1}_{\forall} + \mathrm{M}^1 + \mathrm{KL} + \mathrm{BI}^1 \nvdash \mathrm{S}.$$

- $\mathrm{AC}^{\omega *} := \mathrm{AC!}^{1,\tau} + \mathrm{AC}^{0,\tau}$.
- BI^1 is the restriction of BI^ω to type 1 objects.

Summary.

$$\begin{tikzcd}
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Review.

1. $\mathrm{ACA} \leftrightarrow \Sigma^0_1\text{-LEM} + \Pi^0_1\text{-AC}^{0,0}$.
2. $\mathrm{WKL} \leftrightarrow \Sigma^0_1\text{-DML} + \Pi^0_1\text{-AC}^{\vee}$.

- $\Sigma^0_1\text{-LEM} \leftrightarrow \Pi^0_1\text{-LEM}$ by M^0.

Roughly speaking, our meta-theorem allows one to detect using classical reasoning on $\mathrm{Uni}(\mathrm{S})$ that S implies at least the $\Pi^0_1\text{-LEM}$ rather than only the strictly weaker principle $\Sigma^0_1\text{-DML}$.

Future Work.

- In intuitionistic reverse mathematics, a lot of relationships between non-constructive principles still remain to be open.
- Theorems of this kind might be strong tools to analyze the structure of hierarchy between non-constructive principles for constructive reverse mathematics.

$\Rightarrow$ Analyze relationships between non-constructive principles by using theorems of this kind and uniform reverse mathematics!

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Thank you for your attention!