Point-free characterisation of Bishop compact metric spaces

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Constructive Mathematics

By constructive mathematics I mean a mathematics which

- uses intuitionistic logic.
- is predicative: the class of Pow(X) of the subsets of an inhabited set X is not set.
- accepts some constructive choice principles, e.g. the axiom of Dependent Choice:

  DC. Given a set A, a total relation $R \subseteq A \times A$ and $a_0 \in A$, there exists a function $f\colon \mathbb{N} \to A$ such that $f(0) = a_0$ and for all $n \in \mathbb{N}$, $f(n)Rf(n+1)$.
- is compatible with other schools of constructivism. In particular, Fan theorem is not acceptable.

Topology in Constructive mathematics

Metric space

The theory of metric spaces as described in Bishop's *Foundations of Constructive Analysis* is well established in constructive mathematics. However, its extension to general topology has a major difficulty.

- Most of the compact metric spaces fail to be topologically compact without recourse to Fan theorem.

Point-free topology

A promising approach to general topology (without relying on Fan theorem) is **formal topology** (Sambin, 1987).

- A point-free topology adapted from the theory of locale (frame).
- Many spaces behave better in formal topology. Formal Cantor space and Formal unit interval are topologically compact.
- Successfully constructivised many results of classical topology: Tychonoff's theorem for compact topologies (without choice).

Connection between metric spaces and formal topology

The connection between Bishop's metric space and formal topology has been unclear until recently.

- The compactness in formal topology via open cover and compactness in Bishop's metric space via completeness and totally boundedness.

Theorem (Palmgren, 2007). There exists a full and faithful functor from the category of locally compact metric spaces to the category of locally compact regular formal topologies.

Our work. Characterise the image of the compact metric spaces of the Palmgren's functor in formal topologies – point-free characterisation of compact metric spaces.

Classically, this a special case of Urysohn's metrisation theorem.

Theorem. The following are equivalent for a topological space X:

1. X is second countable and compact Hausdorff.
2. X is can be embedded as a compact subspace of $\prod_{n \in \mathbb{N}} [0, 1]$.
3. X is compact and metrisable.

Formal topology

A **formal topology** $\mathcal{S}$ is a triple $\mathcal{S} = (S, \lhd, \leq)$ where $(S, \leq)$ is a preorder and $\lhd \subseteq S \times \mathrm{Pow}(S)$ is a **covering relation** on S such that

$$\mathcal{A}\,U = \{a \in S \mid a \lhd U\}$$

is a set for each $U \subseteq S$ and that

$$\frac{a \in U}{a \lhd U}, \quad \frac{a \leq b}{a \lhd b}, \quad \frac{a \lhd U \quad U \lhd V}{a \lhd V}, \quad \frac{a \lhd U \quad a \lhd V}{a \lhd U \downarrow V},$$

for all $a, b \in S$ and $U, V \subseteq S$ where

$$U \lhd V \overset{\text{def}}{\iff} (\forall a \in U)\, a \lhd V,$$

$$U \downarrow V \overset{\text{def}}{=} {\downarrow}U \cap {\downarrow}V = \{c \in S \mid (\exists a \in U)\,(\exists b \in V)\, c \leq a \,\&\, c \leq b\}.$$

Example. Let $(X, \mathcal{B})$ be a topological space presented by a family of basic opens $\mathcal{B}$. Then, $(\mathcal{B}, \lhd_X, \leq_X)$ defined by $a \leq_X b \overset{\text{def}}{\iff} a \subseteq b$ and $a \lhd_X U \overset{\text{def}}{\iff} a \subseteq \bigcup U$ is a formal topology.

Formal topology

Let $\mathcal{S} = (S, \lhd, \leq)$ be a formal topology. The operator

$$\mathcal{A} : \text{Pow}(S) \to \text{Pow}(S) : U \mapsto \mathcal{A}\,U = \{a \in S \mid a \lhd U\}$$

is a closure operation on Pow(S). The class of fixed points of the operation, denoted by $Sat(\mathcal{S})$, forms a **frame** (or a complete Heyting algebra), a complete lattice where finite meets distribute over arbitrary joins:

$$\mathcal{A}\,U \wedge \bigvee_{i \in I} \mathcal{A}\,U_i = \bigvee_{i \in I} \mathcal{A}\,U \wedge \mathcal{A}\,U_i$$

for all $U \subseteq S$ and a family of subsets $U_i \subseteq S$ $(i \in I)$.

Formal topology map

Let $\mathcal{S}$ and $\mathcal{S}'$ be formal topologies. A **(formal topology) map** from $\mathcal{S}$ to $\mathcal{S}'$ is a relation $r \subseteq S \times S'$ such that

1. $S \lhd r^{-}S'$,
2. $r^{-}\{a\} \downarrow r^{-}\{b\} \lhd r^{-}(\{a\} \downarrow' \{b\})$,
3. $a \lhd' U \implies r^{-}\{a\} \lhd r^{-}U$

for all $a, b \in S'$ and $U \subseteq S'$. The class $Hom(\mathcal{S}, \mathcal{S}')$ of formal topology maps is equipped with the equality $r = s \overset{\text{def}}{\iff} \mathcal{A}\, r^{-}\{a\} = \mathcal{A}\, s^{-}\{a\}$ $(a \in S')$. A formal topology map $r : \mathcal{S} \to \mathcal{S}'$ induces a frame map

$$\mathcal{A}\, r^{-}(-) : Sat(\mathcal{S}') \to Sat(\mathcal{S}).$$

A **point** of a formal topology $\mathcal{S}$ is a subset $\alpha \subseteq S$ such that

1. $(\exists a \in S)\, a \in \alpha$,
2. $a, b \in \alpha \implies (c \in a \downarrow b)\, c \in \alpha$,
3. $a \lhd U \;\&\; a \in \alpha \implies (\exists b \in U)\, b \in \alpha$.

The collection of points of $\mathcal{S}$ is denoted by $Pt(\mathcal{S})$.

Inductively generated formal topology

Let S be a set. An **axiom-set** on S is a pair (I, C), where $(I(a))_{a \in S}$ is a family of sets, and C is a family $(C(a, i))_{a \in S, i \in I(a)}$ of subsets of S.

Theorem (Coquand, Sambin, Smith, and Valentini, 2003). Let $(S, \leq)$ be a preordered set, and let (I, C) be an axiom-set on S. Then, there exists a covering relation $\lhd_{I,C}$ inductively generated by the following rules:

$$\frac{a \in U}{a \lhd_{I,C} U} \text{ (reflexivity)}, \quad \frac{a \leq b \quad b \lhd_{I,C} U}{a \lhd_{I,C} U} \text{ ($\leq$-left)},$$

$$\frac{a \leq b \quad i \in I(b) \quad a \downarrow C(b, i) \lhd_{I,C} U}{a \lhd_{I,C} U} \text{ ($\leq$-infinity)}.$$

The relation $\lhd_{I,C}$ is the least covering relation on S which satisfies ($\leq$-left) and $a \lhd_{I,C} C(a, i)$ for each $a \in S$ and $i \in I(a)$.

The formal topology $\mathcal{S} = (S, \lhd_{I,C}, \leq)$ together with the axiom set (I, C) which generates $\lhd_{I,C}$ is called an **inductively generated formal topology**. A pair $(a, C(a, i))$ for each $a \in S$ and $i \in I(a)$ is called an axiom of $\mathcal{S}$ and will be written $a \lhd_{I,C} C(a, i)$.

I.g formal topology – Points and Examples

Let $\mathcal{S} = (S, \lhd_{I,C}, \leq)$ be an inductively generated formal topology with an axiom set (I, C). A point of $\mathcal{S}$ is a subset $\alpha \subseteq S$ such that

1. $(\exists a \in S)\, a \in \alpha$,
2. $a, b \in \alpha \implies (c \in a \downarrow b)\, c \in \alpha$,
3. $a \in \alpha \implies (\exists b \in C(a, i))\, b \in \alpha$

for each $a, b \in S$ and $i \in I(a)$.

Formal Cantor space. Let $S = \{0, 1\}^*$ be ordered by $l \leq l' \overset{\text{def}}{\iff} (\exists k \in S)\, l' * k = l$. Formal Cantor space $\mathcal{C}$ is generated by the following axiom-set on S:

$$l \lhd \{l * \langle 0 \rangle, l * \langle 1 \rangle\}$$

Explicitly, define $I(l) = \{*\}$ and $C(l, *) = \{l * \langle 0 \rangle, l * \langle 1 \rangle\}$ for each $l \in S$. We have $Pt(\mathcal{C}) \cong 2^{\mathbb{N}}$.

Examples

Formal Reals. Let $S = \{(p,q) \in \mathbb{Q} \times \mathbb{Q} \mid p < q\}$ be ordered by $(r,s) \leq (p,q) \overset{\text{def}}{\iff} r \leq p \;\&\; q \leq s$. Formal reals $\mathcal{R}$ is generated by the following axiom set on S.

(R1) $(p,q) \lhd_{\mathcal{R}} \{(r,s) \in S \mid p < r < s < q\}$,

(R2) $(p,q) \lhd_{\mathcal{R}} \{(p,s),(r,q)\}$ for each $p < r < s < q$.

We have $Pt(\mathcal{R}) \cong \mathbb{R}$, where $\mathbb{R}$ is the Dedekind cuts.

A formal topology map $r : \mathcal{S}' \to \mathcal{S}$ is an **embedding** if it is (impredicatively) a regular monomorphim. A **subtopology** $\mathcal{S}'$ of a formal topology $\mathcal{S} = (S, \lhd, \leq)$ is the image of an embedding: a subtopology $\mathcal{S}'$ is of form $(S, \lhd', \leq)$ such that $\lhd \subseteq \lhd'$ which implies $Pt(\mathcal{S}') \subseteq Pt(\mathcal{S})$.

Example. **The formal unit interval** $\mathcal{I}[0,1]$ is a subtopology of the formal reals $\mathcal{R}$ determined by the axioms **(R1)** and **(R2)** together with the additional axiom

(R3) $(p,q) \lhd_{\mathcal{I}[0,1]} \{(p,q) \mid p < 1 \;\&\; 0 < q\}$,

for each rational interval (p,q). More axioms implies bigger covering and fewer points. We have $Pt(\mathcal{I}[0,1]) \cong [0,1]$.

Overt formal topology

Let $\mathcal{S}$ be a formal topology. A **positivity predicate** on $\mathcal{S}$ is a subset $Pos \subseteq S$ which satisfies

$$\textbf{(Mon)}\quad a \lhd U \;\&\; Pos(a) \implies (\exists b \in U)\, Pos(b),$$

$$\textbf{(Pos)}\quad a \lhd \{x \in S \mid x = a \;\&\; Pos(a)\}$$

for all $a \in S$, where $Pos(a) \overset{\text{def}}{\iff} a \in Pos$. Intuitively, $Pos(a)$ if "the basic open a is inhabited". Every formal topology admits at most one positivity predicate. A formal topology is **overt** if it is equipped with a positivity predicate.

Example. Formal Cantor space $\mathcal{C}$ and Formal reals $\mathcal{R}$ are overt with $Pos = S$. The formal unit interval $\mathcal{I}[0,1]$ is overt with the positivity

$$Pos = \{(p,q) \in S \mid p < 1 \;\&\; 0 < q\}.$$

Note. Classically, every formal topology is overt. Constructively, overtness is non-trivial.

Localic completion (Vickers, 2005; Palmgren, 2007)

Let $X = (X, \rho)$ be a metric space, and let $\mathbb{Q}^{>0}$ be the set of positive rationals. A **formal ball** $\mathsf{b}(x, \varepsilon)$ is a pair $(x, \varepsilon) \in X \times \mathbb{Q}^{>0}$. We write M_X for $X \times \mathbb{Q}^{>0}$. Define an order $\leq_X$ and a strict order $<_X$ on M_X by

$$\mathsf{b}(x, \delta) \leq_X \mathsf{b}(y, \varepsilon) \overset{\text{def}}{\iff} \rho(x, y) + \delta \leq \varepsilon,$$
$$\mathsf{b}(x, \delta) <_X \mathsf{b}(y, \varepsilon) \overset{\text{def}}{\iff} \rho(x, y) + \delta < \varepsilon.$$

Note. The conditions are not equivalent to the (strict) inclusion of between the actual balls $B(x, \varepsilon) = \{y \in X \mid \rho(x, y) < \varepsilon\}$.

The **localic completion** of a metric space (X, ρ) is a formal topology $\mathcal{M}(X) = (M_X, \lhd_X, \leq_X)$ inductively generated by the following axiom-set on M_X:

(M1) $a \lhd_X \{b \in M_X \mid b <_X a\}$,

(M2) $a \lhd_X \mathcal{C}_\varepsilon$ for each $\varepsilon \in \mathbb{Q}^{>0}$

for all $a \in M_X$, where we define $\mathcal{C}_\varepsilon = \{\mathsf{b}(x, \varepsilon) \in M_X \mid x \in X\}$, the set of formal balls with radius ε.

Localic completion

For any metric space $X = (X, \rho)$

- its localic completion $\mathcal{M}(X)$ is always overt.
- the points $Pt(\mathcal{M}(X))$ is a completion of X: $Pt(\mathcal{M}(X))$ is isometric to the set $\tilde{X}$ of Cauchy sequences on X modulo the standard equality.
- if $Y \subseteq X$ is a dense subset of X, then $\mathcal{M}(Y) \cong \mathcal{M}(X)$.
- $\mathcal{M}(2^{\mathbb{N}}) \cong \mathcal{C}$, $\mathcal{M}(\mathbb{R}) \cong \mathcal{R}$ and $\mathcal{M}([0,1]) \cong \mathcal{I}[0,1]$.

A metric space is **compact** if it is complete and totally bounded.

A formal topology $\mathcal{S}$ is **compact** if

$$S \lhd U \implies (\exists U_0 \in \mathsf{Fin}(U))\, S \lhd U_0$$

for all $U \subseteq S$.

Theorem (Palmgren, 2007). The localic completion $\mathcal{M}$ restricts to a full and faithful functor $\mathcal{M} : \mathbf{Comp} \to \mathbf{KFTop}$, where

$\mathbf{Comp}$ the category of compact metric spaces and uniformly continuous functions.

$\mathbf{KFTop}$ the category of compact formal topologies and maps.

Compact overt sub-topologies a localic completion

Spitters (2010) and Coquand, Palmgren, and Spitters (2011) observed that a compact subspace of a Bishop locally compact metric space gives rise to a compact overt subtopologies of its localic completion, and vise versa.

Theorem. Let $X = (X, \rho)$ be a compact metric space. Then, up to isomorphism, the localic completion $\mathcal{M} \colon \mathbf{Comp} \to \mathcal{M}(\mathbf{Comp})$ induces an order isomorphism between the compact subspaces of X and the compact overt subtopologies of $\mathcal{M}(X)$.

Proof. Given a compact subspace $Y \subseteq X$, its localic completion $\mathcal{M}(Y)$ embeds into $\mathcal{M}(X)$ as an overt compact subtopology.
Conversely, given a compact overt subtopology $\mathcal{S}$ of $\mathcal{M}(X)$, the points $Pt(\mathcal{S})$ is metrically isomorphic to a compact subset of X. $\square$

Corollary. The following are equivalent for a formal topology $\mathcal{S}$.

1. $\mathcal{S}$ is isomorphic to $\mathcal{M}(X)$ of some compact metric space X.
2. $\mathcal{S}$ is isomorphic to a compact overt subtopology of $\mathcal{M}(X)$ of some compact metric space X.

The image of countable products

For any set-indexed family $(\mathcal{S}_i)_{i\in I}$ of inductively generated formal topologies, its product $\prod_{i\in I}\mathcal{S}_i$ can be defined predicatively.

Proposition. Let $(X_n,\rho_n)_{n\in\mathbb{N}}$ be a sequence of compact metric spaces. The canonical map $r\colon \mathcal{M}(\prod_{n\in\mathbb{N}} X_n) \to \prod_{n\in\mathbb{N}}\mathcal{M}(X_n)$ corresponding to the projections $\mathcal{M}(\pi_n)\colon \mathcal{M}(\prod_{n\in\mathbb{N}} X_n) \to \mathcal{M}(X_n)$ $(n\in\mathbb{N})$ is an embedding. Moreover, the image of $\mathcal{M}(\prod_{n\in\mathbb{N}} X_n)$ in $\prod_{n\in\mathbb{N}}\mathcal{M}(X_n)$ is the largest overt subtopology of $\prod_{n\in\mathbb{N}}\mathcal{M}(X_n)$.

$$\begin{tikzcd}
\mathcal{M}(\prod_{n\in\mathbb{N}} X_n) \arrow[r, hook, "r"] \arrow[dr, "\mathcal{M}(\pi_n)"'] & \prod_{n\in\mathbb{N}}\mathcal{M}(X_n) \arrow[d, "p_n"] \\
& \mathcal{M}(X_n)
\end{tikzcd}$$

Example. $\mathcal{M}(\prod_{n\in\mathbb{N}}[0,1])$ is the largest overt subtopology of $\prod_{n\in\mathbb{N}}\mathcal{M}([0,1]) \cong \prod_{n\in\mathbb{N}}\mathcal{I}[0,1]$.

Compact Regular Formal Topology

Let $\mathcal{S}$ be a formal topology, and $U, V \subseteq S$. Define

$$U \lll V \overset{\text{def}}{\iff} S \lhd U^* \cup V$$

where $U^* = \{a \in S \mid a \downarrow U \lhd \emptyset\}$.

A formal topology $\mathcal{S}$ is **regular** if there exists a function $wc\colon S \to \mathrm{Pow}(S)$ such that for all $a \in S$

- $(\forall b \in wc(a))\, \{b\} \lll \{a\}$,
- $a \lhd wc(a)$.

Let $\mathbb{I} = \{q \in \mathbb{Q} \mid 0 \leq q \leq 1\}$. A **scale** from U to V is a family $(U_q)_{q \in \mathbb{I}}$ of subsets of S such that

- $U \lhd U_0$, $U_1 \lhd V$,
- $(\forall p, q \in \mathbb{I})\, p < q \implies U_p \lll U_q$.

A scale $(U_q)_{q \in \mathbb{I}}$ from U to V is **finitary** if $U_q \in \mathrm{Fin}(S)$ for all $q \in \mathbb{I}$. Let

$$U \twoheadleftarrow\!\!\lll_{\mathsf{Fin}} V \overset{\text{def}}{\iff} \text{there exists a finitary scale from } U \text{ to } V.$$

Compact Regular Formal Topology

Proposition. Let $\mathcal{S}$ be a compact regular formal topology. Then, for any $U, V \subseteq S$,

$$U \lll V \implies U \twoheadrightarrow\!\!\!\lll_{\mathsf{Fin}} V.$$

Note. The proof relies on the axiom of Dependent Choice.

Proposition (Johnstone, 1982). Let $\mathcal{S}$ be a formal topology, and let $U, V \subseteq S$. Then, the following are equivalent.

1. There exists a scale from U to V.
2. There exists a formal topology map $r \colon \mathcal{S} \to \mathcal{I}[0,1]$ such that
   - $r^{-}(0,\infty) \downarrow U \lhd \emptyset$,
   - $r^{-}(-\infty,1) \lhd V$.

   where

$$(-\infty, 0) \stackrel{\text{def}}{=} \{(p,q) \in S_{\mathcal{R}} \mid q = 0\}, \quad (1,\infty) \stackrel{\text{def}}{=} \{(p,q) \in S_{\mathcal{R}} \mid p = 1\}.$$

Compact enumerably completely regular formal topology

A compact formal topology $\mathcal{S}$ is **enumerably completely regular** if

- there exists a function $wc\colon S \to \mathsf{Pow}(S)$ which makes $\mathcal{S}$ regular,
- the relation $\overline{wc} = \{(a, b) \in S \times S \mid a \in wc(b)\}$ is countable,
- for each $(a, b) \in \overline{wc}$, there exists a choice of finitary scales from $\{a\}$ to $\{b\}$.

Lemma. The localic completion $\mathcal{M}(X)$ of a compact metric space X is isomorphic to an overt compact enumerably completely regular formal topology.

Point-free characterisation of compact metric spaces

Theorem. Let $\mathcal{S}$ be a formal topology. Then, the following are equivalent:

1. $\mathcal{S}$ is isomorphic to an overt compact enumerably completely regular formal topology.
2. $\mathcal{S}$ is isomorphic to a compact overt subtopology of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$.
3. $\mathcal{S}$ is isomorphic to a localic completion of some compact metric space.

Proof. (3 $\Rightarrow$ 1). The previous Lemma.
(1 $\Rightarrow$ 2). If $\mathcal{S}$ is overt compact enumerably completely regular, then the relation $\overline{wc}$ associated with its function $wc : S \to \mathrm{Pow}(S)$ is countable. Since each $(a,b) \in \overline{wc}$ have a choice of scales, $\overline{wc}$ defines a sequence of maps $\mathcal{S} \to \mathcal{I}[0,1]$, and thus it determines a map $r : \mathcal{S} \to \prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$. Regularity of $\mathcal{S}$ ensures that r is an embedding.
(2 $\Rightarrow$ 3). If $\mathcal{S}$ is an overt compact subtopology of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$, then it is a subtopology of $\mathcal{M}(\prod_{n\in\mathbb{N}}[0,1])$. $\prod_{n\in\mathbb{N}}[0,1]$ is a compact metric space, $\mathcal{S}$ is isomorphic to a localic completion of some compact metric space. $\square$

Classical observations

Classically, the following notions are equivalent:

1. An overt compact enumerably completely regular formal topology.
2. A compact regular frame with a countable base.
3. A countable normal distributive lattice.
4. A second countable compact Hausdorff space.
5. A compact metric space.

- ($1 \Leftrightarrow 2$). The previous theorem + classical logic + impredicativity.
- ($2 \Leftrightarrow 3$). Every compact regular frame can be represented as the ideals of the normal distributive lattice which is freely generated by its base.
- ($2 \Leftrightarrow 4$). Compact regular frames are sober (by Prime Ideal Theorem).
- ($4 \Leftrightarrow 5$). Urysohn's metrisation theorem.

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