A Gap Phenomenon for Schnorr Randomness

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Theme

- Random = incompressible
- Descriptive complexity is one of the measure of randomness.

Kolmogorov Complexity

1. If X is random, then

$$K(X \upharpoonright n) \sim n.$$

2. If X is half-random, then

$$K(X \upharpoonright n) \sim n/2.$$

3. If X is far from random, then

$$K(X \upharpoonright n) \sim K(n) \sim \log n.$$

Random and Non-random

Definition

A set $X \in 2^\omega$ is ML-random if there exists a constant $c \in \omega$ such that

$$K(X \upharpoonright n) \geq n - c$$

for every n.

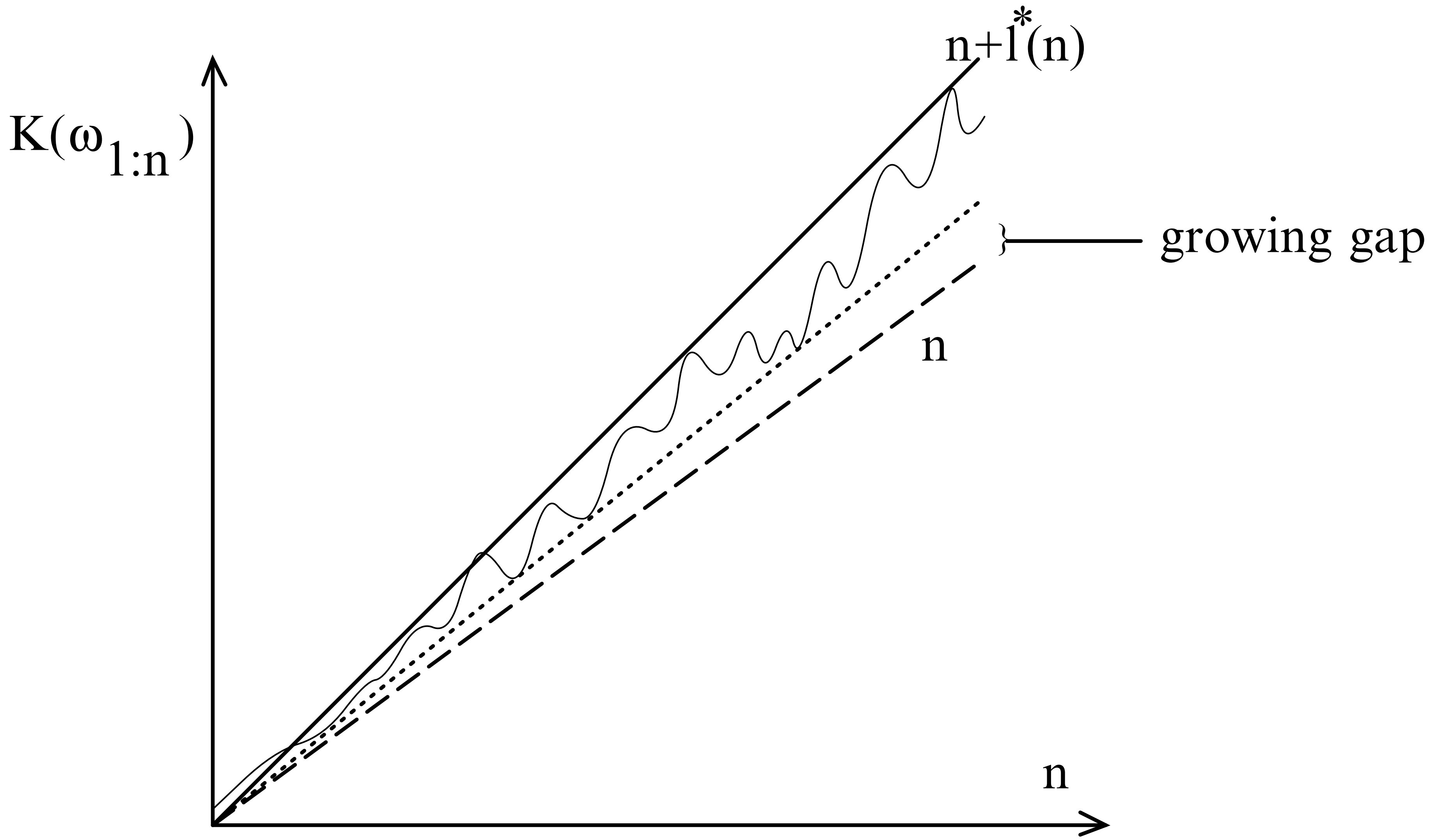
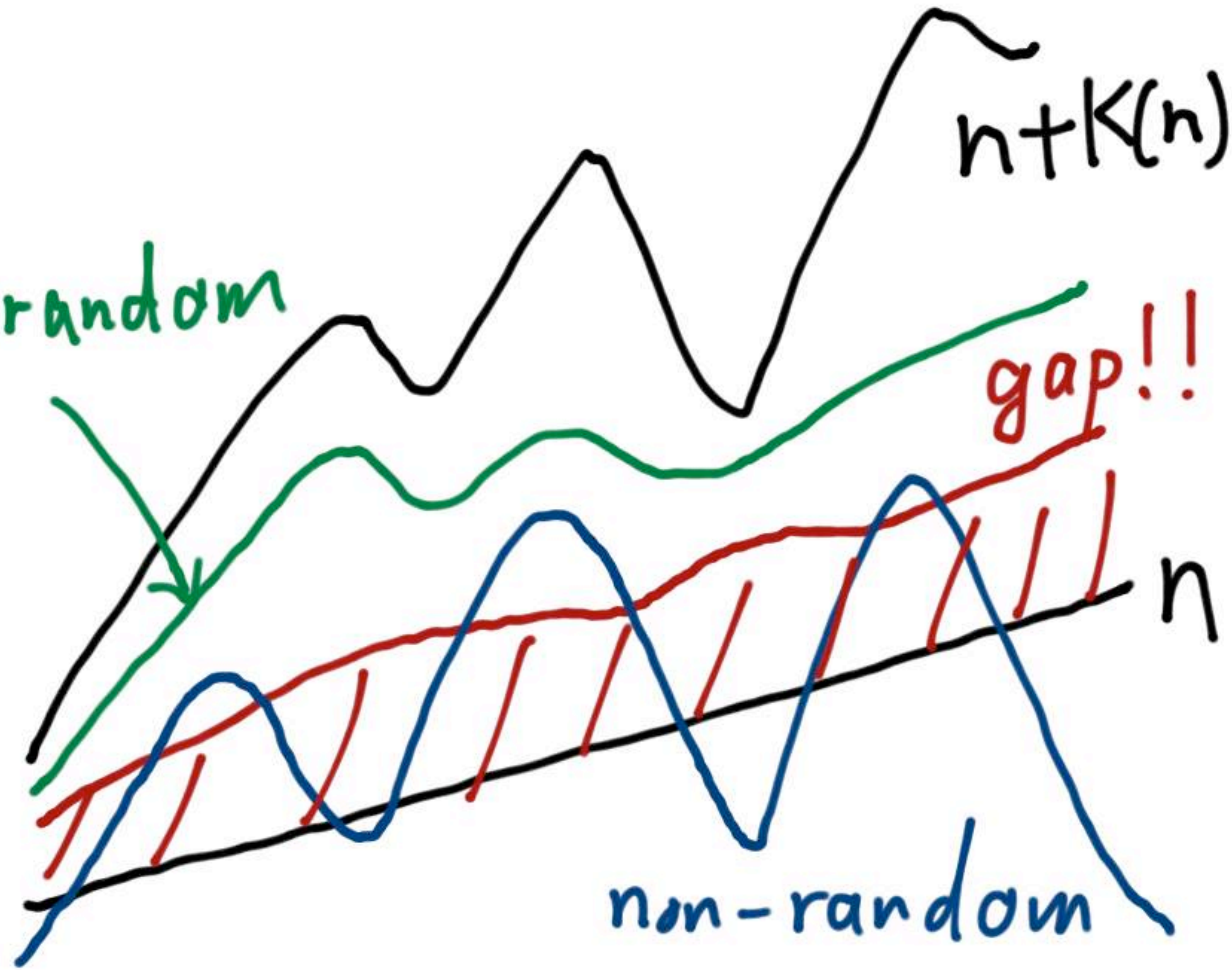


FIGURE 3.3. Complexity oscillations of a typical random sequence ω

From Li and Vitányi (2008) p.224


n+K(n)
random
gap!!
n
non-random

Initial Segment Complexity

Kolmogorov Complexity

The Kolmogorov complexity K of a string σ is defined by

$$K(\sigma) = \min\{|\tau| \ : \ U(\tau) = \sigma\}$$

where U is the prefix-free universal Turing machine.

In other words, $K(\sigma)$ is the length of shortest programs that produces σ.

For each string $\sigma \in 2^{<\omega}$, we have

$$K(\sigma) \leq |\sigma| + K(|\sigma|) + O(1).$$

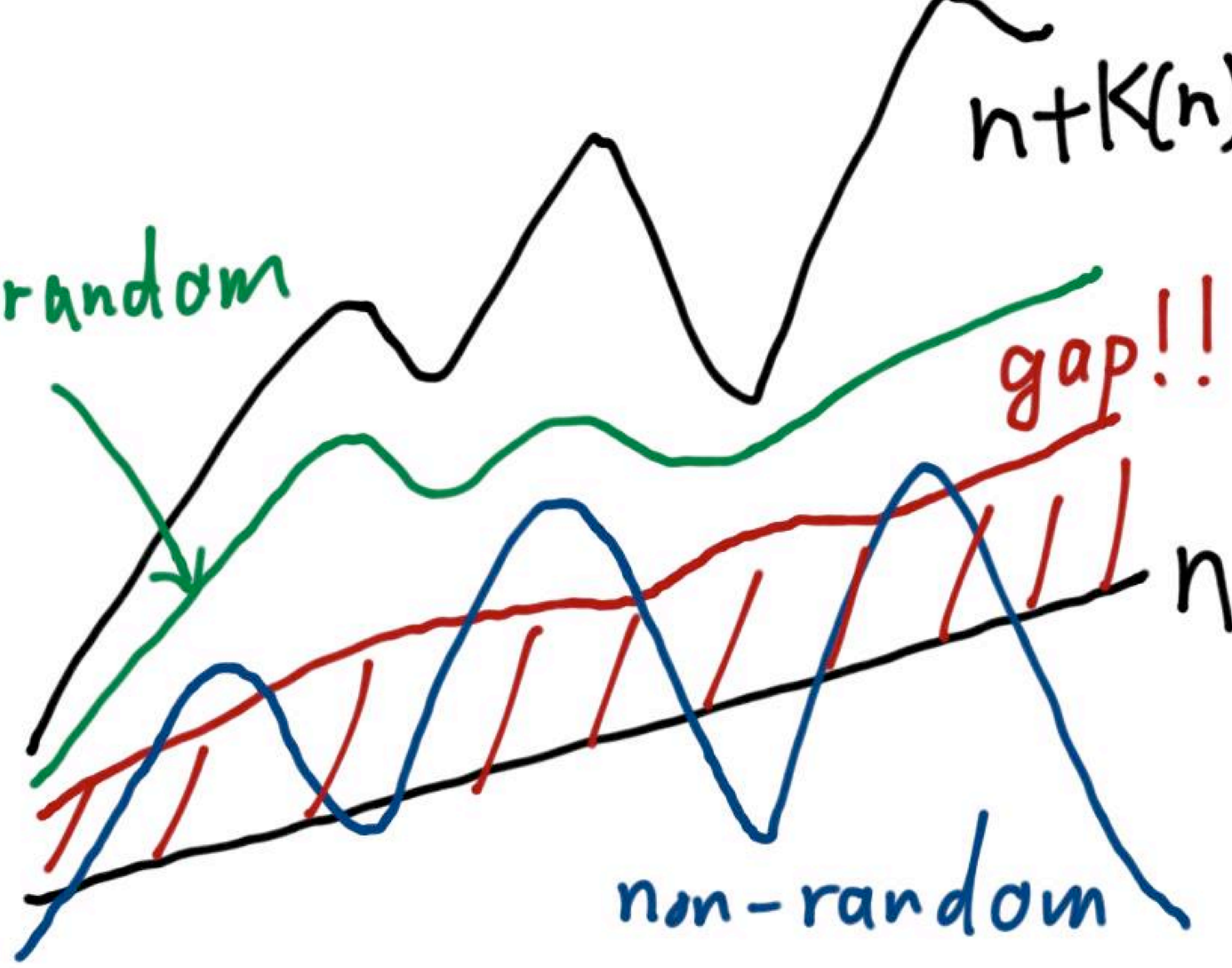
No set $X \in 2^{\omega}$ satisfies with

$$K(X \upharpoonright n) \geq n + K(n) - O(1).$$

The class of the sets satisfying

$$K(X \upharpoonright n) \geq n - O(1)$$

has the measure 1. Such a set is called a ML-random set.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Theorem (Ample Excess Lemma; Miller and Yu 2008)

A set $X \in 2^\omega$ is ML-random if and only if

$$\sum_n 2^{n-K(X\restriction n)} < \infty.$$

Corollary

Let X be a ML-random set. Then,

$$K(X \restriction n) \geq n + K^X(n) - O(1).$$

An important theorem with many applications!!

Schnorr Randomness

A machine is a partial computable function $M :\subseteq 2^{<\omega} \to 2^{<\omega}$. The measure of a machine is

$$\Omega_M = \sum_{\sigma \in \mathrm{dom}(M)} 2^{-|\sigma|}.$$

A machine with a computable measure is called a computable measure machine. A set X is called Schnorr random if

$$K_M(X \restriction n) > n - O(1)$$

for every computable measure machine.

Question

Does a Schnorr version of Ample Excess Lemma hold?

A variant of Omega

Let

$$\widehat{\Omega}_M = \sum_{\sigma \in 2^{<\omega}} 2^{-K_M(\sigma)}$$

Recall that Ω_U is ML-random when U is a prefix-free universal Turing machine. Chaitin has observed $\widehat{\Omega}_U$ is also ML-random.

If M is a computable measure machine (which means that Ω_M is computable), then $\widehat{\Omega}_M$ is also computable.

Extended Ample Excess Lemma

Lemma (M.)

For a machine M, let $f_M : 2^\omega \to \mathbb{R}$ be the function such that

$$f_M(X) = \sum_{n=0}^{\infty} 2^{n - K_M(X \restriction n)}.$$

Then, we have

$$\int f_M(X)\, d\mu = \widehat{\Omega}_M.$$

In particular, if M is a computable measure machine, f_M is a Schnorr integral test.

Recall that

$$f_M(X) = \sum_{n=0}^{\infty} 2^{n-K_M(X\restriction n)}.$$

Then

$$\begin{aligned}\int f_M(X)\ d\mu &= \int \sum_{n=0}^{\infty} 2^{n-K_M(X\restriction n)}\ d\mu \\ &= \sum_{n=0}^{\infty} \sum_{\sigma\in 2^n} 2^{n-K_M(\sigma)} \cdot 2^{-n} \\ &= \sum_{\sigma\in 2^{<\omega}} 2^{-K_M(\sigma)} \\ &= \widehat{\Omega}_M\end{aligned}$$

.

Proposition (M.)

Let X be a Schnorr random set. For every computable measure machine M, there exists a uniformly computable measure machine N such that

$$K_M(X \upharpoonright n) \geq n + K_{N^X}(n) - O(1).$$

Lemma (M.-Rute 2013)

Let t be a Schnorr integral test. Then there is a sequence $\{h_n\}$ of uniformly computable total functions $h_n : 2^\omega \to [0, \infty)$ such that

1. $h_n \leq t$ everywhere,
2. if X is Schnorr random, then there is some n such that $h_n(X) = t(X)$.

Theorem (Miller and Yu 2008)

$X \oplus Z$ is ML-random iff $K(X \upharpoonright (Z \upharpoonright n)) \geq (Z \upharpoonright n) + n - O(1)$.

Theorem (M.)

$X \oplus Z$ is Schnorr random iff $K_M(X \upharpoonright (Z \upharpoonright n)) \geq (Z \upharpoonright n) + n - O(1)$ for every computable measure machine M.

Theorem (Miller and Yu 2008)

Let Z be ML-random. The following are equivalent.

1. $X \oplus Z$ is ML-random.
2. $C(X \upharpoonright n) + K(X \upharpoonright n) \geq 2n - O(1)$.

Theorem (M.)

Let Z be Schnorr random. The following are equivalent.

1. $X \oplus Z$ is Schnorr-random.
2. $C_N(X \upharpoonright n) + K_M(X \upharpoonright n) \geq 2n - O(1)$ for every computable measure machine M and every decidable machine N.

- The results by Miller and Yu (2008) was used to show the relation between K, C and vL-reducibility.
- Similarly, the results presented here provide its Schnorr version; the relation between Schnorr, dm (decidable machine) and vLS-reducibility.

Theorem (Miller 2009)

A set $X \in 2^\omega$ is 2-random if and only if

$$K(X \upharpoonright n) \geq n + K(n) - O(1)$$

for infinitely many n.

Theorem (M.)

For a computable measure machine M, there exists a computable measure machine N such that, for every Schnorr random set X,

$$K_M(X \upharpoonright n) \geq n + K_N(n) - O(1)$$

for infinitely many n.

Summary

- We looked at initial segment complexity of Schnorr random set.
- This requires uniform relativization and computable analysis.
- More relation with truth-table reduction.