Ramseyan factorization theorem in reverse mathematics

Shota Murakami
(Joint work with Takeshi Yamazaki and Keita Yokoyama)

Tohoku University

Contents

Contents

What is Ramseyan factorization theorem?

- We can find it in some books/papers in Automata Theory.
- It is a Ramsey-type theorem about infinite sequences.

What is Ramseyan factorization theorem? (2)

Let A and B be sets. We call A the set of letters and B the set of colors. We also call sequences of elements from A words.

Then the **Ramseyan factorization theorem for A and B** is the following statement:

Definition (Ramseyan factorization theorem RF^A_B)

For every coloring of finite words $f : A^{<N} \to B$ and every infinite word $u \in A^{\mathbb{N}}$, we can cut u and divide it into infinitely many finite words $v_0, v_1, v_2, \ldots$, i.e. $u = v_0 ^\frown v_1 ^\frown v_2 ^\frown \cdots$, and every segment of u of the form $v_i ^\frown v_{i+1} \cdots ^\frown v_j$ $(i \leq j,\ i \geq 1)$ is colored the same by f.

Here, we call such $v = \langle v_i \mid i \in \mathbb{N} \rangle$ a **Ramseyan factorization for f and u.**

What is Ramseyan factorization theorem? (3)

Definition (Ramseyan factorization theorem RF^A_B)

For every coloring of finite words $f : A^{<N} \to B$ and every infinite word $u \in A^{\mathbb{N}}$, we can cut u and divide it into infinitely many finite words $v_0, v_1, v_2, \dots$, i.e. $u = v_0 ^\frown v_1 ^\frown v_2 ^\frown \cdots$, and every segment of u of the form $v_i ^\frown v_{i+1} \cdots ^\frown v_j$ $(i \le j,\ i \ge 1)$ is colored the same by f.

Note: RT^n_k: n denotes which tuples we consider (i.e. n-tuples).
RF^s_k: s denotes the number of letters.

Example

Define $f : \{0,1\}^{<\mathbb{N}} \to \{0,1\}$ and $u \in \{0,1\}^{\mathbb{N}}$ as
$f(\sigma) = $ (the first number of σ) and $u = 101001 \dots 10^i 10^{i+1} 1 \dots$.
Then $v = \langle 0^i 1 \mid i \in \mathbb{N} \rangle$ is a Ramseyan factorization for f and u.

Results about RF

We can prove the following:

Theorem (RCA_0)

$\mathrm{RT}^2_2 \Leftrightarrow \mathrm{RF}^{\mathbb{N}}_k \Leftrightarrow \mathrm{RF}^2_k$ *for all* $k \in \omega,\ k \geq 2$.

In order to study the strength of RF^1_k, we consider the following variant of Ramsey's theorem.

Definition (RT^f_k)

Let $f : [\mathbb{N}]^n \to \mathbb{N}$. *Then* RT^f_k *is the following statement:*
$\forall P : \mathbb{N} \to k \exists H \subseteq_{\text{inf}} \mathbb{N} \forall u, v \in [H]^n \quad P(f(u)) = P(f(v))$.

Results about RF. (2)

Let $\mathrm{Subt}(a, b) = b - a$. Then we can prove the following:

Proposition (RCA_0)

$\mathrm{RT}^{\mathrm{Subt}}_k \Leftrightarrow \mathrm{RF}^1_k$ *for all* $k \in \mathbb{N}$.

Corollary (RCA_0)

$\mathrm{RF}^1_k \Rightarrow \mathrm{RT}^1_k$. *Thus* $\forall k \mathrm{RF}^1_k \Rightarrow \mathrm{B}\Sigma^0_2$.

Contents

Weak Ramseyan factorization

Weak Ramseyan factorization theorem is the following:

Definition (Weak Ramseyan factorization theorem WRF^A_B)

For every coloring of finite words $f : A^{<N} \to B$ and every infinite word $u \in A^{\mathbb{N}}$, we can cut u and divide it into infinitely many finite words $v_0, v_1, v_2, \ldots$, i.e. $u = v_0^\frown v_1^\frown v_2^\frown \cdots$, and every segment of u of the form v_i $(i \geq 1)$ is colored the same by f.

Recall:

Definition (Ramseyan factorization theorem RF^A_B)

For every coloring of finite words $f : A^{<N} \to B$ and every infinite word $u \in A^{\mathbb{N}}$, we can cut u and divide it into infinitely many finite words $v_0, v_1, v_2, \ldots$, i.e. $u = v_0^\frown v_1^\frown v_2^\frown \cdots$, and every segment of u of the form $v_i^\frown v_{i+1}^\frown \cdots ^\frown v_j$ $(i \leq j,\ i \geq 1)$ is colored the same by f.

To study the strength of WRF^s_k, we also consider a weaker Ramsey's theorem as follows:

Definition (Weak Ramsey's theorem WRT^n_k)

For every coloring $P : [\mathbb{N}]^n \to k$, there exists an infinite $H = \{a_0 < a_1 < \cdots\}$ such that every n-tuples of the form $\langle a_i, a_{i+1}, \ldots, a_{i+n-1}\rangle$ $(i \in \mathbb{N})$ is colored the same by P.

Here, such H is called **weak homogeneous** for P.

Note: Subsets of weak homogeneous sets might not be weak homogeneous again.

Thus, the following question is not easy to solve:

Question

Does WRT^n_k imply WRT^n_{k+1} over RCA_0?

Results about WRF and WRT

We can prove the following:

Proposition (RCA_0)

$\mathrm{WRF}^{\mathbb{N}}_k \Leftrightarrow \mathrm{WRT}^2_k$ *for all* $k \in \mathbb{N}$.
In particular, $\mathrm{WRF}^{\mathbb{N}}_2$ *is equivalent to* WRT^2_2.

Recall:

Theorem (RCA_0)

$\mathrm{RT}^2_2 \Leftrightarrow \mathrm{RF}^{\mathbb{N}}_k \Leftrightarrow \mathrm{RF}^2_k$ *for all* $k \in \omega, \ k \geq 2$.

Question

Is WRF^2_k equivalent to WRT^2_k over RCA_0 for all $k \in \mathbb{N}$?

Results about WRF and WRT. (2)

To see the strength of $\mathrm{WRF}^{\mathbb{N}}_2$, or equivalently WRT^2_2, the notion of transitive/semi-transitive colorings is helpful.

Definition (Hirschfeldt/Shore (2007))

A coloring $P : [\mathbb{N}]^2 \to k$ is **transitive** *iff*
$P(a, b) = P(b, c) = i \Rightarrow P(a, c) = i$.
A coloring $P : [\mathbb{N}]^2 \to k$ is **semi-transitive** *iff*
$P(a, b) = P(b, c) = i > 0 \Rightarrow P(a, c) = i$.

Definition (Hirschfeldt/Shore (2007))

Transitive Ramsey's theorem (trRT^2_k)*: Any transitive coloring $P : [\mathbb{N}]^2 \to k$ has an infinite homogeneous set.*
Semi-transitive Ramsey's theorem (strRT^2_k)*: Any Semi-transitive coloring $P : [\mathbb{N}]^2 \to k$ has an infinite homogeneous set.*

Results about WRF and WRT. (3)

Definition

Semi weak Ramsey's theorem (sWRT^2_k): *Any coloring $P : [\mathbb{N}]^2 \to k$ has an infinite homogeneous set H s.t. $P([H]^2) = \{0\}$ or a weak homogeneous set $H = \{a_0 < a_1 < \cdots\}$ s.t. $P(a_0, a_1) > 0$.*

We can prove the following:

Theorem (RCA_0)

$\mathrm{WRT}^2_k \Rightarrow \mathrm{trRT}^2_k$.

Theorem (RCA_0)

$\mathrm{sWRT}^2_2 \Leftrightarrow \mathrm{strRT}^2_2$.

Results about WRF and WRT. (4)

Corollary (RCA_0)

$\mathrm{ADS} \leq \mathrm{WRF}^{\mathbb{N}}_2 = \mathrm{WRT}^2_2 \leq \mathrm{CAC}$.

Proof. By Hirschfeldt/Shore(2007), $\mathrm{ADS} \Leftrightarrow \mathrm{trRT}^2_2$ and $\mathrm{CAC} \Leftrightarrow \mathrm{strRT}^2_2$ over RCA_0. □

Question

Is WRT^2_2 equivalent to ADS or CAC over RCA_0?

Ramseyan factorization for trees

Definition

For given tress $T, S \subseteq 2^{<\mathbb{N}}$, a tree embedding is an injective $\pi : S \to T$ such that for any $\sigma, \tau \in S$, $\pi(\sigma) \cap \pi(\tau) = \pi(\sigma \cap \tau)$. For given a tree embedding $\pi : S \to T$, and for any $\sigma, \tau \in S$ such that $\sigma \subsetneq \tau$, the edge between $\pi(\sigma)$ and $\pi(\tau)$, denoted by $E_\pi(\sigma, \tau)$, is the sequence $\rho \in 2^{<\mathbb{N}}$ such that $\pi(\sigma)^\frown\rho = \pi(\tau)$.

Definition (TRF^2_k)

Ramseyan factorization theorem for trees *(TRF^2_k):*

> *For any infinite tree $T \subseteq 2^{<\mathbb{N}}$ and a coloring $f : 2^{<\mathbb{N}} \to k$, there exists an infinite tree $S \subseteq 2^{<\mathbb{N}}$ and a tree embedding $\pi : S \to T$ such that for any $\sigma \subsetneq \tau \in S$ and $\sigma' \subsetneq \tau' \in S$, $f(E_\pi(\sigma, \tau)) = f(E_\pi(\sigma', \tau'))$.*

Ramseyan factorization for trees (2)

We can prove the following:

Proposition (RCA_0)

TRF^2_k *implies* RF^2_k *for all* $k \in \mathbb{N}$*. In particular,* TRF^2_2 *implies* RF^2_2 *(and, equivalently,* RT^2_2*).*

Theorem (RCA_0)

$\mathrm{WKL}_0 + \mathrm{RT}^2_2 \Rightarrow \mathrm{TRF}^2_2 \Rightarrow \mathrm{RT}^2_2$.

Question

Does TRF^2_2 imply WKL_0 over RCA_0?

Contents

Summary

Over RCA_0,
The strength of RF^s_k is as follows:

$$\mathrm{RT}^2_2 = \mathrm{RF}^{\mathbb{N}}_2 = \mathrm{RF}^2_2.$$
$$\forall k \mathrm{RF}^1_k = \forall k \mathrm{RT}^{\mathrm{Subt}}_k \geq \mathrm{B}\Sigma^0_2.$$

The strength of WRF^s_k is as follows:

$$\cdots \geq \mathrm{WRT}^2_3 = \mathrm{WRF}^{\mathbb{N}}_3 \geq \mathrm{WRT}^2_2 = \mathrm{WRF}^{\mathbb{N}}_2 \geq \mathrm{WRF}^2_2.$$
$$\mathrm{CAC} \geq \mathrm{WRT}^2_2 = \mathrm{WRF}^{\mathbb{N}}_2 \geq \mathrm{ADS}.$$

The strength of TRF^2_k is as follows:

$$\mathrm{WKL}_0 + \mathrm{RT}^2_2 \geq \mathrm{TRF}^2_2 \geq \mathrm{RT}^2_2.$$

References I

[1] Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman.
On the strength of Ramsey's theorem for pairs.
Journal of Symbolic Logic, 66(1):1–55, 2001.

[2] Denis R. Hirschfeldt.
Slicing the truth: On the computability theoretic and reverse mathematical analysis of combinatorial principles.
to appear.

[3] Denis R. Hirschfeldt and Richard A. Shore.
Combinatorial principles weaker than Ramsey's theorem for pairs.
Journal of Symbolic Logic, 72(1):171–206, 2007.

References II

[4] Dominique Perrin and Jean-Éric Pin.
Infinite Words: Automata, Semigroups, Logic and Games, volume 141.
Academic Press, 2004.

[5] Stephen G. Simpson.
Subsystems of Second Order Arithmetic.
Perspectives in Mathematical Logic. Springer-Verlag, 1999. XIV + 445 pages; Second Edition, Perspectives in Logic, Association for Symbolic Logic, Cambridge University Press, 2009, XVI+ 444 pages.