Comparing sets of natural numbers using randomness and lowness properties.

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Motivation

- One way to classify $\mathcal{P}(\mathbb{N})$ is to define a reducibility and a degree structure.
- In fact, many structures studied in recursion theory such as structures, equivalence relations, mass problems, real life problems (complexity theory), etc is commonly compared this way.
- A reducibility is usually a pre-ordering used to compare the "strength" of two reals.
  - When one problem is harder to solve than another (mass problems, complexity theory)
  - When information given about one real naturally produces information about the other ($\leq_T$, $\leq_e$)
  - When one real contains more "information" than another ($\leq_{LR}$, $\leq_K$, etc)

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- Reducibilities are used to define when a real is weak in information content (which we denote generically as “low"), and its dual “highness".

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Classical Reducibilities

- Most classical reducibilities are defined in terms of an underlying (usually continuous) map that induces the reduction, e.g.

  $A \leq_T B$ iff there is a computable continuous functional
  $$\Phi : \mathcal{P}(\mathbb{N}) \mapsto \mathcal{P}(\mathbb{N}) \text{ such that } \Phi(A) = B.$$

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Reducibilities using Randomness

- The study of relative randomness lead to new reducibilities being looked at. (e.g. Downey-Hirschfeldt-Laforte, Nies).
- In fact, Nies has explicitly listed some conditions which a preordering $\leq_W$ should have to be considered a weak reducibility:
  - It should be weaker than Turing reducibility (used as the benchmark in recursion theory), i.e. for all sets A, B,

$$A \leq_T B \implies A \leq_W B$$

  - The reducibility should be easily definable, i.e. $\leq_W$ should be Σ^0_n as a relation on sets.
  - $X' \not\leq_W X$ for any X.

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- So a weak reducibility should not be too different from the Turing reducibility.
- E.g.

$$A \leq_{ar} B \Leftrightarrow A \leq_T B^{(n)} \text{ for some } n$$

  should not be considered a weak reducibility.
- If $A \leq_W B$ then B can only understand a small part or aspect of A. Compare to $A \leq_T B$ where B knows everything of A.
- Weak reducibilities usually do not have an underlying map which induces the reduction.
  - Σ^0_3 so each reduction still has an index.
  - However each reduction might reduce many (even uncountably many) reals B to a single one A, i.e. $B \leq_W A$.

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Weak Reducibilities

- Some considerations. Given a real,
  - How random is it compared to another?
  - How much information is contained in its initial segments?
  - How much power does it have to compress finite binary strings?
  - How much power does it have to derandomnize other reals?
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Reducibilities using Randomness

- A list of the more common weak reducibilities:

$A \leq_T B$	the benchmark
$A \leq_{LK} B$	$K^B(\sigma) \leq^+ K^A(\sigma)$
$A \leq_{LR} B$	every B random is A-random
$A \leq_{JT} B$	Every partial A-recursive function can be traced by a B-r.e. trace

- Miller shows that $\leq_{LK} \quad = \quad \leq_{LR}$.
- In this talk we will focus on the last two reducibilities.

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Other weak reducibilities

- There are many other weak reducibilities studied (See Nies's book).

$$
\begin{aligned}
A \leq B &\iff A' \leq_T B' \\
A \leq_{CT} B &\iff A \text{ is computably traceable relative } B \\
A \leq_{cdom} B &\iff \text{each } A\text{-recursive function is} \\
& \text{ dominated by a } B\text{-recursive function.} \\
A \leq_{SJT} B &\iff A \text{ is strongly jump traceable by } B \\
& \text{ (a partial relativization).}
\end{aligned}
$$

Some other ones, which are not weak reducibilities:

$$
\begin{aligned}
A \leq_{rK} B &\iff \exists c \forall n (\ K(A \upharpoonright n \mid K(B \upharpoonright n) \leq c\ ) \\
A \leq_{K} B &\iff K(A \upharpoonright n) \leq^{+} K(B \upharpoonright n) \\
A \leq_{C} B &\iff C(A \upharpoonright n) \leq^{+} C(B \upharpoonright n)
\end{aligned}
$$

Work on weak reducibilities

- There is a large literature on work regarding these weak reducibilities. Some questions which have been considered include:
  - For which sets A is the lower cone $\{B : B \leq_W A\}$ countable?
  - Is every set A bounded (in the sense of $\leq_W$) by a 1-random?
  - Are the 1-randoms closed upwards under $\leq_W$?
  - Which sets are W-complete (or W-hard)? That is, for which sets A is $A \geq_W \emptyset'$?
  - Since $\equiv_W$ is weaker than $\equiv_T$, the structure of Turing degrees within a single W-degree.
  - What can be said about the degree structure of $\equiv_W$?
- One approach not well-studied in the literature is the concept of a W-base for randomness. This will be our concern in this talk for $W = LR, JT$.

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LR and JT-reducibilities

- We focus on these two reducibilities.

Definition (JT-reducibility, due to Simpson)

- A B-trace with bound h is a uniformly B-c.e. sequence V_n^B such that for every n, $\#V_n^B \leq h(n)$.
- We say that a B-trace V_n^B traces a partial function ψ if for every n, $\psi(n)\downarrow \Rightarrow \psi(n) \in V_n^B$.
- $A \leq_{JT} B$ iff every partial A-recursive function ψ^A is traced by some B-trace with a computable bound h.

- In particular $A \leq_{JT} \emptyset$ means that A is jump traceable.
- $\emptyset' \leq_{JT} A$ means that A is JT-hard.
  (Simpson) If A is Δ_2^0 this is equivalent to A being superhigh.

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LR and *JT*-reducibilities

Definition (*LR*-reducibility)

We say that $A \leq_{LR} B$ iff every B-random set is A-random.

- In particular $A \leq_{LR} \emptyset$ means that A is K-trivial.
- (Kjos-Hanssen, Miller, Solomon) $\emptyset' \leq_{LR} A$ means that A is uniformly almost everywhere dominating.

Lemma

$A \leq_{LR} B \;\Rightarrow\; A \leq_{JT} B$

- This is done by observing that the proof of “low for random implies jump traceable" relativizes correctly (using a characterization of Kjos-Hanssen, Miller, Solomon).

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Using weak reducibilities to define lowness

- A “lowness property" is a property asserting that a given set A resembles $\emptyset$ in some way.
- Many of the weak reducibilities are the result of relativizing a certain lowness property arising in randomness. E.g.

$$\leq_{LK}, \quad \leq_{LR}, \quad \leq_{JT}, \quad \leq_{SJT}, \quad \leq_{CT}, \quad \leq_{cdom}.$$

- So $A \leq_W \emptyset$ means that A is low in the sense of W.

Computed by many sets

- Another interpretation of “A is low" is that A is easy to compute.

Theorem (Sacks)

A is non-recursive iff $\{Z : Z \geq_T A\}$ is null.

- So nullness is too coarse. What if we change null to effectively null in A?

Definition (Kučera)

A is a (Turing) base for randomness if $A \leq_T Z$ for some A-random Z.

- So being *not* a base for randomness means that $\{Z : Z \geq_T A\}$ can be described by an A-effectively null set (in the sense of ML-tests).

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If A a base for randomness then A is low for K.

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- The weak reducibilities play a role here. It has not been explored fully.

Definition

For a weak reducibility $\leq_W$, we say that A is a *W-base for randomness* if $A \leq_W Z$ for some A-random set Z.

- These properties mean that A is easy to compute in the sense of $\leq_W$. Trivially,
  - Each K-trivial set is low for random and hence an LR-base for randomness.
  - Each jump traceable set is a JT-base for randomness.
- But are these two notions trivial? Do you get more?

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JT-base is trivial

Theorem (Franklin-N-Solomon)

Each JT-base for randomness is jump traceable.

(Hence this notion is trivial).

Proof.

Similar to the "Hungry Sets Theorem" of Hirschfeldt-Nies-Stephan.

- Suppose ψ^A is traced by T^B for some A-random set B. We wish to build an unrelativized c.e. trace V for ψ^A.
- If we see $\psi^\sigma(x)\downarrow$ we want to obtain assurance that σ is a possible initial segment of A.
- To do this we issue descriptions of all reals Z such that T_x^Z contains the value $\psi^\sigma(x)$.

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JT-base is trivial

Proof continued.

- We keep "eating" these strings Z until we have described 2^{-x} much reals Z.
- Only after we have eaten 2^{-x} much reals Z do we finally believe that $\sigma \subset A$ could be correct, and enumerate $\psi^{\sigma}(x)$ into the unrelativized trace V_x.
- Note that if $\sigma \subset A$ was *really the case*, then we must be able to eat up at least 2^{-x} much Z and so $\psi^{A}(x)$ will be traced in V_x.

JT-base is trivial

Proof continued.

- Now what is the size of V_x?
- For each value $\psi^\sigma(x)$ that we believe and enumerate in V_x, there is a corresponding 2^{-x} much measure of oracles Z such that $T_x^Z \ni \psi^\sigma(x)$.
- How many different values $\psi^\sigma(x)$ can we do this?
- At most $2^x \cdot t(x)$, where $t(x)$ is the computable bound for $\# T_x^B$.
- So $\# V_x \leq 2^x \cdot t(x)$.

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□

LR-bases

- For LR-bases the situation is a lot more interesting. We know that LR-bases are strictly larger than the class of K-trivial reals.

Proposition

There exists an LR-base A which is low for Ω but not K-trivial.

Proof.

Barmpalias, Lewis and Stephan constructed a Π^0_1-class P where every path is LR-reducible to Ω and not K-trivial. Apply the low-for-Ω basis theorem to P. □

- Since this example gives a LR-base A which is not Δ^0_2, it is natural to ask if

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- The answer is also no, provided by indirect means. We will come back to this.
- First, observe that LR-bases are closed downwards under $\leq_{LR}$:

  If $A \leq_{LR} B \leq_{LR} Z$ for some B-random Z, then surely Z is also A-random.
- (C. Porter) If $A \leq_{LR} X, Y$ where X and Y are relatively random, then A is an LR-base.

  Since X is Y-random and $A \leq_{LR} Y$, so X is also A-random.

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- (Barmpalias) Every LR-base A is generalized low (i.e. $A' \leq_T A \oplus \emptyset'$).
- Every LR-base is a JT-base. Hence every LR-base is in fact jump traceable.
- If we restrict our study further to the LR-bases which are r.e., we get interestingly

$$K\text{-trivial} \subsetneq LR\text{-base} \subsetneq \text{superlow}.$$

No other randomness class is known to lie strictly in between.

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K-trivial $\subsetneq$ LR-base $\subsetneq$ superlow.

- By examining the previous proof, each LR-base is jump traceable with bound $h(n) = 2^n$. So not every superlow c.e. set is an LR-base.

Proposition (C. Porter)

There exists an r.e. set A which is an LR-base and not K-trivial.

Proof.

Barmpalias showed that if X and Y are Δ^0_2 sets such that $X, Y >_{LR} \emptyset$, then there is a c.e. set A such that

$$\emptyset <_{LR} A \leq_{LR} X, Y.$$

Take X, Y to be Δ^0_2 relatively random sets. Then A is an LR-base. $\square$

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Take X, Y to be Δ^0_2 relatively random sets. Then A is an LR-base. $\square$

LR-bases

- Downey and Greenberg showed that each $\sqrt{\log n}$-jump traceable c.e. set is K-trivial. So we get for c.e. sets,

$$\sqrt{\log n}\text{-jump traceable} \subsetneq LR\text{-base} \subseteq 2^n\text{-jump traceable.}$$

Question

For which computable functions h are h-jump traceable sets an LR-base?

- This question follows similar attempts at characterizing K-triviality in terms of traceability.

LR-bases

Theorem (Franklin-N-Solomon)

There is a c.e. set A which is an LR-base such that A is not jump traceable with the identity bound.

Proof.

- We present a direct construction of a c.e. set A such that A is an LR-base but is not K-trivial.
- To make A an LR-base, we build a c.e. operator V and a set B such that $U^A \subseteq V^B$ where U^A is the universal A-c.e. set of strings of measure < 1 and $\mu(V^B) < 1$.

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There is a c.e. set A which is an LR-base such that A is not jump traceable with the identity bound.

Proof.

- We present a direct construction of a c.e. set A such that A is an LR-base but is not K-trivial.
- To make A an LR-base, we build a c.e. operator V and a set B such that $U^A \subseteq V^B$ where U^A is the universal A-c.e. set of strings of measure < 1 and $\mu(V^B) < 1$.

LR-bases

Continued.

- To make B random relative to A, we ensure that $B \notin [T^A]$ where T is some component of the universal ML-test relative A with small measure.
- To make A K-trivial, we try and make $U^A \not\subseteq E$ where E is a c.e. set of strings with $\mu(E) < 1$. Let's look at one such positive requirement.
- This positive requirement acts by enumerating a string σ into U^α and V^β (where α, β are current approximations to A and B). We must do this because we need to ensure $U^A \subseteq V^B$.

LR-bases

Continued.

- One of three things can happen:
  - (I) $[\beta] \subseteq T^\alpha$. Then this β cannot be used anymore as B must be made A-random. We move to another β' and enumerate σ in $V^{\beta'}$, until $\mu(T^\alpha) > 2^{-t}$ (for some threshold t).
  - (II) σ enters E
  - (III) Nothing ever happens.
- If nothing ever happens, then we would have met the positive requirement (as $\sigma \in U^A - E$).
- If (I) happens first then we abandon this cycle by restraining A and forbidding all the 2^{-t} much strings $\beta \in T^\alpha$ from being used as B again.
- Note that we abandon this cycle in (I) only at most 2^t times, as each time we restrain A increasing the measure of T^A by 2^{-t}.

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LR-bases

Continued.

- Finally if (II) happens first then we change α to a new one (which allows us to clear σ from U^A while σ is permanently stuck in E).
- We would however also have lost some measure in V because we have enumerated σ into V^β for at most 2^{-t} much β which is now wasted. However the average measure lost in V is less than $2^{-t} \cdot 2^{-|\sigma|}$ while the opponent has lost $2^{-|\sigma|}$ in E (a lot more than us).
- At the end, since $\mu(E) < 1$, this single requirement will produce wastage in V of at most 2^{-t} (which can be made arbitrarily small).

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$\square$

More questions

Question

- *Is there a Δ^0_2 LR-base which is not superlow? Such an LR-base must be low.*
- *What is the quantity of LR-bases? Is there a perfect Π^0_1 class containing only LR-bases?*
- *Is there a non-recursive hyperimmune-free LR-base? What about computably traceable?*

- Thank you.

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