Connecting the provable with the unprovable

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Overview

- Introduction
- Preliminaries
- Two examples: the Paris–Harrington theorem and the adjacent Ramsey theorem
- A phase transition with two parameters

Introduction

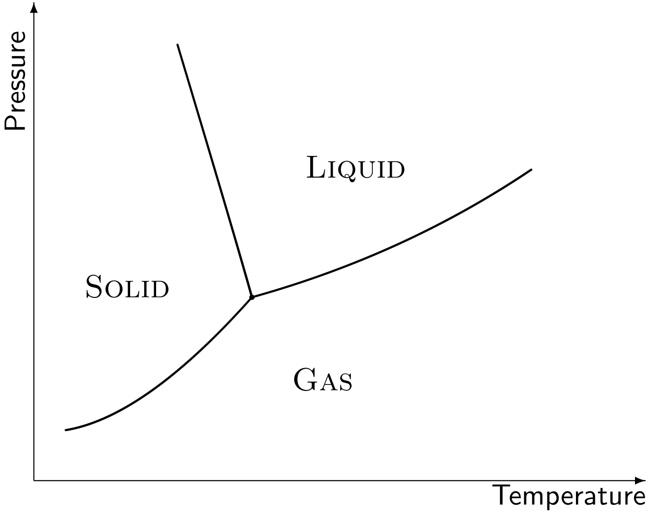

Figure: *Phase transitions in physics*

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An important feature in each of these theorems is that one can point at a part of the statement which 'makes the theorem unprovable'. For example the largeness condition in PH or the limited condition for colourings in AR.

This condition is dependent on some element under consideration, for example the minimal element of a homogeneous set in PH or the maximum of the input in AR. Hence one can introduce a parameter function $f\colon \mathbb{N} \to \mathbb{N}$ at this place in the theorem ψ to obtain theorems ψ_f.

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The original theorem ψ is ψ_{id}. If, for constant function c, ψ_c is provable then the obvious question to ask is where between[1] the identity and constant functions ψ_f changes from provable to unprovable: *the transition threshold*.

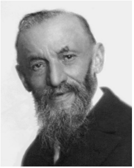
[1]functions ordered by eventual domination

A programme was started by Andreas Weiermann to classify these transitions. Results from this programme include transitions for the Paris–Harrington, Kanamori–McAloom theorems, Dickson's and Higman's lemma.

The goal of studying phase transitions in logic is to better understand independence.

Preliminaries

Peano Arithmetic



Giuseppe Peano

(1858-1932)

PA consists of:

- The basic axioms, which define successor, addition and multiplication.
- Induction axiom scheme.

Fragments of PA

Induction scheme:

$$[\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1))] \rightarrow \forall x \varphi(x)$$

When the induction scheme is restricted to Σ_n-formulas the resulting theory is called $\mathrm{I}\Sigma_n$.

Ordinals below ε_0

$$0, 1, 2, 3, 4, 5, 6, \ldots, \omega,$$
$$\omega + 1, \omega + 2, \omega + 3, \ldots, \omega + \omega = \omega \cdot 2, \ldots, \omega \cdot \omega = \omega^2,$$
$$\omega^2 + 1, \ldots, \omega^\omega = \omega_2, \ldots, \omega^{\omega_2} = \omega_3, \ldots, \omega_\omega = \varepsilon_0.$$

Ordinals: Cantor Normal Forms

All $\alpha < \varepsilon_0$ can be written uniquely in the Cantor Normal Form:

$$\alpha = \omega^{\alpha_1} \cdot m_1 + \cdots + \omega^{\alpha_n} \cdot m_n,$$

where $\alpha_1 > \cdots > \alpha_n$ and $m_1 > 0, \ldots, m_n > 0$, $n \geq 1$.

Ordinals: Fundamental sequences

$$
\begin{aligned}
(\alpha + 1)[x] &= \alpha, \\
(\alpha + \omega^{\alpha_n + 1} \cdot (m+1))[x] &= \alpha + \omega^{\alpha_n + 1} \cdot m + \omega^{\alpha_n} \cdot x, \\
(\alpha + \omega^{\gamma} \cdot (m+1))[x] &= \alpha + \omega^{\gamma} \cdot m + \omega^{\gamma[x]}.
\end{aligned}
$$

Hardy hierarchy

$$\begin{aligned} H_0(i) &= i, \\ H_{\alpha+1}(i) &= H_\alpha(i+1), \\ H_\gamma(i) &= H_{\gamma[i]}(i+1). \end{aligned}$$

Two examples of phase transitions

Paris–Harrington

Theorem (PH^d_f, Paris and Harrington, 1977)

For every r, m there exists an R such that for every colouring $C\colon [m, R]^d \to r$ there exists an $H \subseteq [m, R]$ of size at least $f(\min H)$ for which C limited to $[H]^d$ is constant.

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Theorem (Weiermann, 2003)

1. $\mathrm{I}\Sigma_d \nvdash \mathrm{PH}^{d+1}_{\frac{\log^d}{c}}$.
2. $\mathrm{I}\Sigma_1 \vdash \mathrm{PH}^{d+1}_{\log^{d+1}}$.

Adjacent Ramsey

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Theorem (AR^d_f, Friedman, 2010)

For every r there exists R such that for every f-limited colouring $C\colon \{0,\dots,R\}^k \to \mathbb{N}^r$ there are $x_1 < \cdots < x_{d+1} \leq R$ with $C(x_1,\dots,x_d) \leq C(x_2,\dots,x_{d+1})$.

Adjacent Ramsey

Theorem

1. $\mathrm{I}\Sigma_{d+1} \nvdash \mathrm{AR}^{d+1}_{\sqrt[c]{\log^d}}$.
2. $\mathrm{I}\Sigma_1 \vdash \mathrm{AR}^{d+1}_{\log^{d+1}}$.

Remark

These transitions correspond to estimates on the functions

$$k \mapsto \mathrm{PH}_k^d(m, r).$$

and

$$k \mapsto \mathrm{AR}_k^d(r).$$

The inverses of the lower bound estimates are parameter values for which the theorems remain independent. Inverses of upper bound estimates are parameter values for which the theorems are provable.

A phase transition with two parameters

Theorem ($\mathrm{ARPH}^d_{f,g}$)

For every m, r there exists R such that for all f-limited $C\colon \{m, \dots, R\}^d \to \mathbb{N}^r$ there exist $x_1 < \cdots < x_{g(x_1)}$ for which $C(x_1, \dots, x_d) \leq C(x_2, \dots, x_{d+1}) \leq \cdots \leq C(x_{g(x_1)-d}, \dots, x_{g(x_1)})$.

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Theorem

- $\mathrm{I}\Sigma_{d+1} \vdash \mathrm{ARPH}^{d+1}_{c,\mathrm{id}}$.
- $\mathrm{I}\Sigma_{d+1} \nvdash \mathrm{ARPH}^{d+1}_{\mathrm{id},\mathrm{id}}$.

Theorem

$$\mathrm{I}\Sigma_{d+1} \vdash \mathrm{ARPH}^{d+1}_{H_\alpha^{-1},\mathrm{id}} \Leftrightarrow \alpha \geq \omega_{d+1}$$

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- $\mathrm{I}\Sigma_{d+1} \nvdash \mathrm{ARPH}^{d+1}_{\sqrt[c]{\log^d},c}$.
- $\mathrm{I}\Sigma_{d+1} \vdash \mathrm{ARPH}^{d+1}_{\log^{d+1},c}$.

Concluding remark

The transitions for $\mathrm{ARPH}^{d}_{f,g}$ have been classified for g equal to the constant function and for g equal to the identity function. What happens between those two parameter values for g?

Concluding remark

The expectation is:

If $g \leq \log^{d-1}$ then $\mathrm{ARPH}^d_{f,g}$ behaves as $\mathrm{ARPH}^d_{f,c}$.

If $g \geq \sqrt[c]{\log^{d-2}}$ then $\mathrm{ARPH}^d_{f,g}$ behaves as $\mathrm{ARPH}^d_{f,\mathrm{id}}$.

Thank you for listening.

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