Blackwell Games with a Constraint Function

Weiguang Peng

Mathematical Institute, Tohoku University
Joint work with Takeshi Yamazaki

February 19, 2014

Contents

An Example

Let's consider the "Rock-Paper-Scissor" game.

A **play** of this game can be depicted as follows:

Player I :	*Rock*	*Paper*	*Rock*	...
Player II :	*Rock*	*Paper*	*Rock*	...

In this game, if one of them wins, the play stops and

- the pay-off of player I is 1 if player I wins,
- the pay-off of player I is 0 if player II wins,
- the pay-off of player I is $\frac{1}{2}$ if they draw.

The contents of interest

Lower value and upper value:

- Lower value: The smallest upper bound on the payoff that player I can guarantee.
- Upper value: The largest lower bound on the restrictions player II can put on player I's payoff.

The lower value of "Rock-Paper-Scissor": $\frac{1}{3}$,

The upper value of "Rock-Paper-Scissor": $\frac{1}{3}$.

Determinacy:

- The lower value equals the upper value.

Blackwell Games

Here, we introduce Blackwell games in a slight extended form.

Definition

A **Blackwell game** is a tuple $B = ((X_i)_{i\in\mathbb{N}}, (Y_i)_{i\in\mathbb{N}}, f)$, where

- X_i's and Y_i's are finite nonempty sets,
- $f : W \to [0,1]$ is a function called payoff function, where $W = \prod_{i=0}^{\infty}(X_i \times Y_i)$.

A play $w = < (x_0, y_0), (x_1, y_1), \cdots >$ of B can be depicted as follows:

$$\begin{array}{lcccccc} \text{Player I}: & x_0 & x_1 & x_2 & \dots & x_i & \dots \\ \text{Player II}: & y_0 & y_1 & y_2 & \dots & y_i & \dots \end{array}$$

In this game, for each round $i \in \mathbb{N}$, player I selects $x_i \in X_i$ and player II selects $y_i \in Y_i$ simultaneously. We call $f(w)$ the **pay-off** of player I for a play w.

If $X_i = X$ and $Y_i = Y$ for each i, we denote $B = ((X_i)_{i\in\mathbb{N}}, (Y_i)_{i\in\mathbb{N}}, f)$ by (X, Y, f).

Notations

- p is a finite prefix of a play, called a position.
- P denotes the set of all positions.
- $[p]$ denotes the set $\{w \mid p \prec w\}$ of all plays hitting p.
- $p|_i$ denotes the prefix of p of length i.
- $\mathcal{D}(X)$ is the set of all probability distributions on X.

Strategy and Probability measure

Definition (Mixed strategy)

A *strategy* for player I is a function σ assigning to each position $p \in \bigcup_{n=0}^{\infty} \prod_{i<n}(X_i \times Y_i)$ a probability distribution on $X_{lh(p)}$.

Similarly, we can define the mixed strategy τ for player II.

Definition (Probability measure)

For a strategy profile (σ, τ) of B, $\mu_{\sigma,\tau}$ denotes the *probability measure* on W determined by

$$\mu_{\sigma,\tau}([p]) = \prod_{i<n}\Big(\sigma(p|_i)(x_i) \cdot \tau(p|_i)(y_i)\Big),$$

for any position $p =< (x_0, y_0), \ldots, (x_{n-1}, y_{n-1}) >\in P$.

Lower value and upper value

If player I plays according to σ and player II plays according to τ, we can define the **lower expected pay-off** of player I at σ, τ:

$$E^{-}_{\sigma,\tau}(f) = \sup_{g \le f,\ g:\text{measurable}} \int g(w) d\mu_{\sigma,\tau}(w),$$

and **upper expected pay-off** at σ, τ:

$$E^{+}_{\sigma,\tau}(f) = \inf_{g \ge f,\ g:\text{measurable}} \int g(w) d\mu_{\sigma,\tau}(w).$$

Lower value and **upper value** are defined as follows:

$$val^{\downarrow}(B) = \sup_{\sigma} \inf_{\tau} E^{-}_{\sigma,\tau}(f);$$

$$val^{\uparrow}(B) = \inf_{\tau} \sup_{\sigma} E^{+}_{\sigma,\tau}(f).$$

Determinacy results

A Blackwell game B is **determined** if $\mathrm{val}^{\downarrow}(B) = \mathrm{val}^{\uparrow}(B)$.

Theorem (AD $\Rightarrow$ AD-BL, Martin 98)

Assuming the Axiom of Determinacy, every Blackwell game is determined.

Corollary (ZFC)

Every Borel Blackwell game is determined.

Blackwell games with constraints

Let $B = (X, Y, f)$ be a Blackwell game.

- S_I denotes the set of strategies for player I,
- and S_{II} denotes the set of strategies for player II.

Definition

If $C_I \subseteq S_I$ and $C_{II} \subseteq S_{II}$ are nonempty, we call $B = (X, Y, f)$ with C_I and C_{II} the Blackwell game with constraints, say $B(f; C_I, C_{II})$.

With respect to ordinary Blackwell games, we can also define the *lower and upper* values of Blackwell games with constraints as follows:

$$\mathrm{val}^{\downarrow}(B(f; C_I, C_{II})) = \sup_{\sigma \in C_I} \inf_{\tau \in C_{II}} E_{\sigma,\tau}(f)$$

$$\mathrm{val}^{\uparrow}(B(f; C_I, C_{II})) = \inf_{\tau \in C_{II}} \sup_{\sigma \in C_I} E_{\sigma,\tau}(f)$$

A Blackwell game with constraints $B(f; C_I, C_{II})$ is **determined** if $\mathrm{val}^{\downarrow}(B(f; C_I, C_{II})) = \mathrm{val}^{\uparrow}(B(f; C_I, C_{II}))$.

Fact

Even for finite sets C_I and C_{II}, there exists a case $\mathrm{val}^{\uparrow}(B(f; C_I, C_{II})) \neq \mathrm{val}^{\downarrow}(B(f; C_I, C_{II}))$.

We would like to know when the Blackwell game with constraints is determined.

Blackwell games with a Constraint Function

Definition

A Blackwell game with a constraint function is a quintuple $B = (X, Y, f, g, m)$, where

- (X, Y, f) is Blackwell game,
- $g : \mathbb{N} \to (X \cup Y \to [0, \infty)$ is called a constraint function,
- $m > 0$.

$B = (X, Y, f, g, m)$ is the same as $B(f; C_I^{g,m}, C_{II}^{g,m})$ where

- $C_I^{g,m} = \{\sigma \in S_I : \forall p, \sum_{x \in X} g(lh(p))(x)\sigma(p)(x) \leq m\}$,
- $C_{II}^{g,m} = \{\tau \in S_{II} : \forall p, \sum_{y \in Y} g(lh(p))(y)\tau(p)(y) \leq m\}$.

Determinacy result

Definition

A Blackwell game with a constraint function is **determined** if

$$(C_I^{g,m} \neq \phi \textit{ and } C_{II}^{g,m} \neq \phi) \Rightarrow$$

$$\sup_{\sigma \in C_I^{g,m}} \inf_{\tau \in C_{II}^{g,m}} E^-_{\sigma,\tau}(f) = \inf_{\tau \in C_{II}^{g,m}} \sup_{\sigma \in C_I^{g,m}} E^+_{\sigma,\tau}(f).$$

Theorem (Main Theorem)

Assuming AD, every Blackwell game with a constraint function is determined.

Theorem (ZFC)

Every Borel Blackwell game with a constraint function is determined.

Idea of Proof. We will only show the case of one-round games. Let $B = (\{a_1, \cdots, a_k\}, \{b_1, \cdots, b_l\}, f, g, m)$ be a one-round Blackwell game with a constraint function. In this case, each strategy for player I and II can be regarded as an element of $[0,1]^k$ and $[0,1]^l$, resp.

We can construct a one-round Blackwell game $B' = (X', Y', F)$ as follows: Take X' and Y' as

- $\{\sum_{a \in X'} r_a a \in [0,1]^k : 0 \leq r_a \text{ and } \sum_{a \in X'} r_a = 1\} = C_I^{g,m}$.
- $\{\sum_{b \in Y'} r_b b \in [0,1]^l : 0 \leq r_b \text{ and } \sum_{b \in Y'} r_b = 1\} = C_{II}^{g,m}$.
- X' and Y' consists of affinely independent elements, that is, the above representations of strategies are unique.

Define a function $F : X' \times Y' \to [0,1]$ by $F(a,b) = \int f \, d\mu_{a,b}$.

Then, we get a bijective transformation from strategies σ, τ of B to strategies σ' and τ' of B' such that

$$E_{\sigma',\tau'}(F) = E_{\sigma,\tau}(f).$$

So, $\sup_{\sigma \in C_I^{g,m}} \inf_{\tau \in C_{II}^{g,m}} E_{\sigma,\tau}(f) = \sup_{\sigma'} \inf_{\tau'} E_{\sigma',\tau'}(F)$ etc.
By the determinacy of Borel Blackwell games, B is also determined.

$\square$

Example

Let $B = (X, Y, f, g, m)$ be a Blackwell game with a constraint function, where $X = \{x_0, x_1\}$, $Y = \{y_0, y_1\}$ and $m = 2$. If we assign the constraint function as $g(x_0) = g(y_0) = g(y_1) = 1$ and $g(x_1) = 3$.

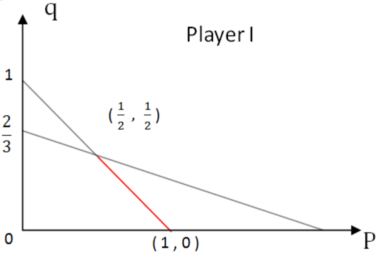
How to construct B':
For the given two player's strategies σ and τ, by the assignment of constraint function, then

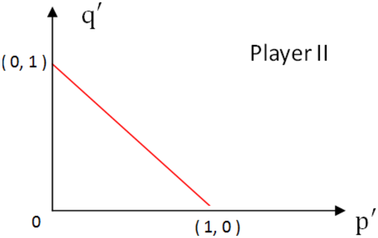
$$(p, q) \in C_I^{g,2} \Longleftrightarrow (p, q) \in S_I \textit{ and } p + 3q \leq 2$$

and also

$$(p', q') \in C_{II}^{g,2} \Longleftrightarrow (p', q') \in S_2 \textit{ and } p' + q' = 1.$$

Thus, $C_I^{g,2}$ and $C_{II}^{g,2}$ are shown by the red segments in the following figures.





We now construct a *one-round* Blackwell game $B' = (X', Y', F)$ as follows: $X' = \{(1,0), (\frac{1}{2}, \frac{1}{2})\}$, and $Y' = \{(1,0), (0,1)\}$ such that

$$F(w) = \begin{cases} f(x_0, y_0) & \text{if } w = ((1,0),(1,0)), \\ f(x_0, y_1) & \text{if } w = ((1,0),(0,1)), \\ \frac{1}{2}f(x_0,y_0) + \frac{1}{2}f(x_1,y_0) & \text{if } w = (\frac{1}{2}, \frac{1}{2}),(1,0)), \\ \frac{1}{2}f(x_0,y_1) + \frac{1}{2}f(x_1,y_1) & \text{if } w = (\frac{1}{2}, \frac{1}{2}),(0,1)). \end{cases} \quad (1)$$

Thank you very much!

References

1 Martin, D. A.: The determinacy of Blackwell games Logic. Vol. 63, No. 4, pp. 1565-1581, 1998.