Gap-sequences

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Joint work with Michael Rathjen and Andreas Weiermann

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Introduction:
The theorem of Higman

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Formally,

$$\begin{aligned} (a_1, \ldots, a_n) \leq^* (b_1, \ldots, b_m) & \\ \iff & a_1 \leq b_{i_1}, \\ & \ldots \\ & a_n \leq b_{i_n}, \end{aligned}$$

for certain $1 \leq i_1 < i_2 < \cdots < i_n \leq m$.

The theorem of Higman

This ordering can be generalized to an arbitrary partial ordering.

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Theorem (Higman)

If X is a well-partial-ordering, then X^* is also a well-partial-ordering.

Important theorem in well-partial-ordering theory!

What is a $\mathbb{wpo}$?

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A well-partial-ordering ($\mathbb{wpo}$) is a partial ordering that is

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- has no infinite antichain.

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- well-founded,
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Definition

A **well-partial-ordering** $(X, \leq_X)$ is a partial ordering such that for every infinite sequence $x_1, x_2, \ldots$ of elements in X, indices $i < j$ exists such that $x_i \leq_X x_j$.

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$$(a_1^1, \ldots, a_{n_1}^1), \ldots, (a_1^k, \ldots, a_{n_k}^k), \ldots, (a_1^l, \ldots, a_{n_l}^l), \ldots$$

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Introduction:
Maximal order type

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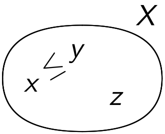

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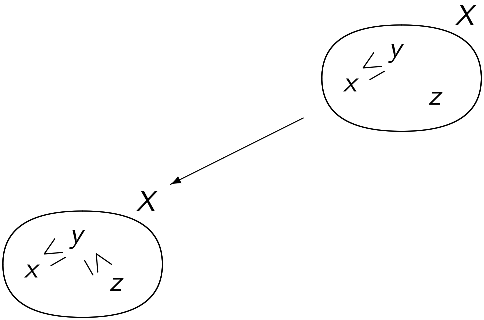

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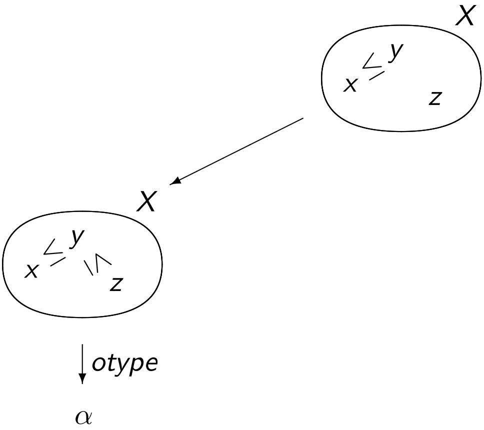

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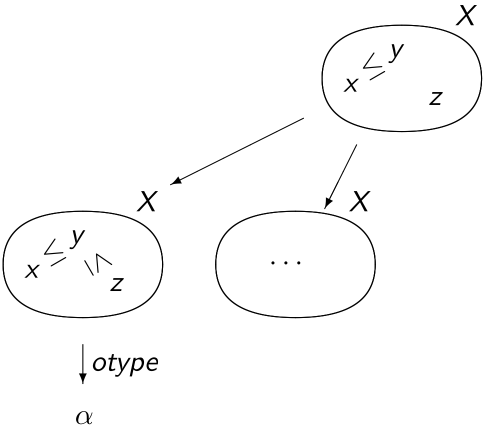

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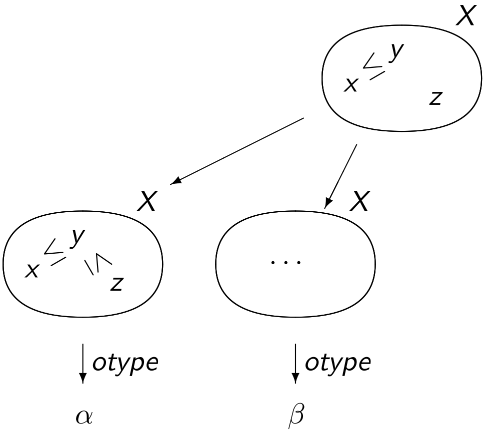

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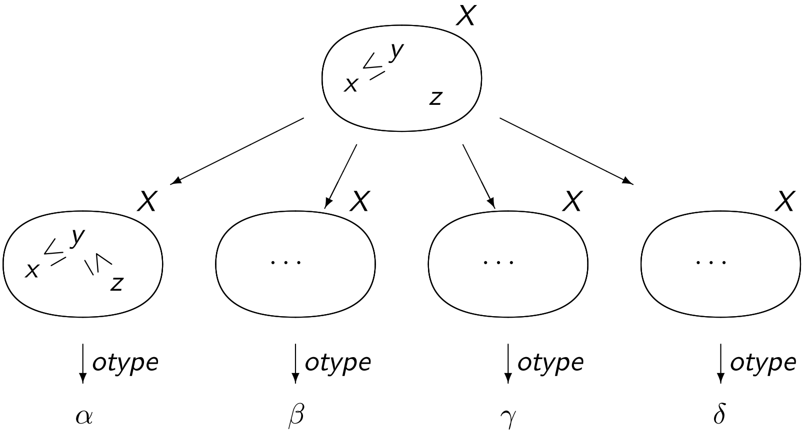

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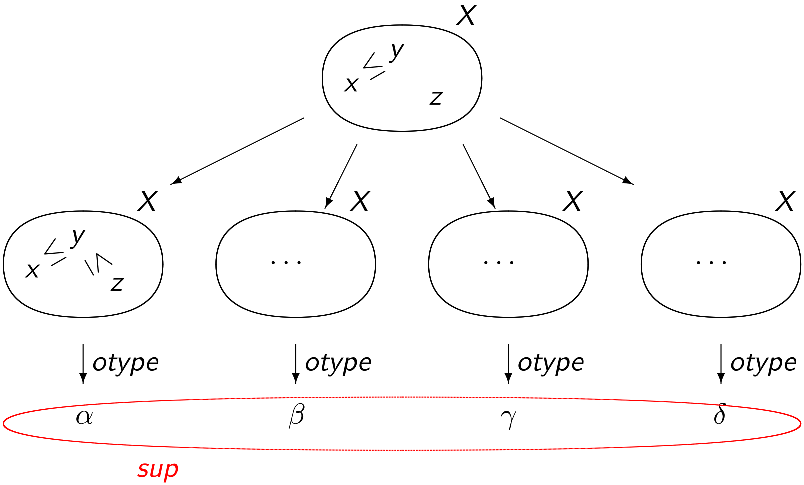

Higman ordering

Theorem (De Jongh & Parikh; D. Schmidt)

If X is a well-partial-ordering, then

$$o(X^*) = \begin{cases} \omega^{\omega^{o(X)+1}} & \text{if } o(X) \text{ is equal to } e+n \\ & \qquad \text{with } e \text{ an epsilon number and } n < \omega, \\ \omega^{\omega^{o(X)-1}} & \text{if } o(X) \text{ is finite,} \\ \omega^{\omega^{o(X)}} & \text{otherwise.} \end{cases}$$

Gap-sequences: Definition

Theorem (Higman)

If X is a $\mathbb{wpo}$*, then* $(X^*, \leq^*)$ *is a* $\mathbb{wpo}$*.*

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Theorem

Let $S = \mathbb{N}^*$*, then* $(S, \leq^*)$ *is a* $\mathbb{wpo}$*.*

$$o(S, \leq^*) = \omega^{\omega^\omega}.$$

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$o(S, \leq^*) = \omega^{\omega^\omega}$.

Now, define a different kind of ordering on $S \rightarrow$ the gap-ordering.

Adding more strength to the statement!

Gap-sequences: definition

$$0120 \qquad \leq \qquad 12022222120$$

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$$0120 \qquad \leq \qquad 120\underbrace{2222}_{\geq}120$$

$$1$$

Gap-sequences: definition

Definition

$(a_1, \ldots, a_n) \leq^w_{gap} (b_1, \ldots, b_m)$

$\Leftrightarrow$

$\exists 1 \leq i_1 < \cdots < i_n \leq m$ such that $a_j = b_{i_j}$ for all j, and

for all k, if $i_1 < k < i_2$ then $b_k \geq b_{i_2}$,

$\ldots$

for all k, if $i_{m-1} < k < i_m$ then $b_k \geq b_{i_m}$.

Gap-sequences: definition

Definition

$(a_1, \ldots, a_n) \leq^s_{gap} (b_1, \ldots, b_m)$

$\Leftrightarrow$

$\exists 1 \leq i_1 < \cdots < i_n \leq m$ such that $a_j = b_{i_j}$ for all j, and

for all k, if $k < i_1$, then $b_k \geq b_{i_1}$,

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- Linearized version of the gap-trees.

Theorem (Friedman, 1982)

Π^1_1–$CA_0 \nvdash \forall n < \omega$ ' $\mathbb{T}^{gap}_n$ *is a wpo'.*

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Theorem (Friedman, 1982)

Π^1_1–$CA_0 \nvdash \forall n < \omega$ '$\mathbb{T}^{gap}_n$ *is a wpo*'.

- Studied in 1985 by Schütte and Simpson

Gap-sequences: Already known results

Results of Schütte-Simpson (1985)

Definition

S_n is the set of finite sequences of natural numbers $< n$.

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For every n, $(S_n, \leq^w_{gap})$ and $(S_n, \leq^s_{gap})$ are $\mathbb{wpo}$'s.

Denote them as S^w_n and S^s_n

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They also have results on the maximal order types!

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$o(S^s_{n+1}) = o(S^s_n \times (S^s_n)^*)$

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Theorem

$o(S^w_{n+1}) = o(\overline{S^s_{n+1}})$, *where* $\overline{S^s_{n+1}}$ *are the sequences* **starting with zero** *with the strong gap-embeddability relation.*

Hence,

Theorem

$o(S^w_{n+1}) = o((S^s_n)^*)$.

Gap-sequences and the theta-functions

Question more precisely

$$0110 \leq^{w}_{gap} 01110$$

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$0110 \leq^{w}_{gap} 01110 \longrightarrow \theta_0\theta_1\theta_1(0) \leq \theta_0\theta_1\theta_1\theta_1(0)$

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$0110 \leq^{w}_{gap} 01110$ $\longrightarrow$ $\theta_0\theta_1\theta_1(0) \leq \theta_0\theta_1\theta_1\theta_1(0)$

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wpo	wo

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$$S_2^{gap,w} \qquad \text{terms of } \theta_0, \theta_1 = T$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\mathbb{wpo} \qquad\qquad \mathbb{wo}$$

So T is a *linear extension* of the gap-sequences!

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$S_2^{gap,w}$ $\downarrow$ wpo | terms of $\theta_0, \theta_1 = T$ $\downarrow$ wo

So T is a *linear extension* of the gap-sequences!
Is it a **maximal linear extension**? Meaning:
is $otype(T) = mot(S_2^{gap,w})$?

T is the set of the well-defined theta-terms, meaning

Definition

1. $0 \in T$ and $S(0) = 0$,
2. $\alpha \in T$, then $\theta_i \alpha \in T$.

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However, $\theta_0(\theta_2(0))$ is not defined.

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Definition

1. $0 \in T$ and $S(0) = 0$
2. $\alpha \in T$, $S(\alpha) \leq i + 1$, then $\theta_i\alpha \in T$ and $S(\theta_i\alpha) := i$

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Set of 'coefficients':

Definition

1. $K_i(0) := 0$
2. $K_j(\theta_i\alpha) = \begin{cases} \theta_i\alpha & \text{if } i \leq j, \\ K_j(\alpha) & \text{if } i > j. \end{cases}$

$K_1(\theta_2\theta_3\theta_2\theta_0\theta_1(0)) = \theta_0\theta_1(0)$.

We define the following ordering on it

Definition

1. $0 < \alpha$, for all $\alpha \neq 0$
2. $i < j$ implies $\theta_i(\alpha) < \theta_j(\beta)$
3. $$\left.\begin{array}{c} \alpha < \beta \text{ and } K_i(\alpha) < \theta_i(\beta) \\ \text{or} \\ \alpha > \beta \text{ and } \theta_i(\alpha) \leq K_i(\beta) \end{array}\right\} \Rightarrow \theta_i(\alpha) < \theta_i(\beta)$$

Back to our question: gap-sequences with **two** labels

Are the theta-functions a maximal linear extension of the gap-sequences on two labels?
Meaning:

$$\begin{aligned} \text{Is} \quad & \sup_{n_1,\dots,n_k} \theta_0\theta_1^{n_1}\dots\theta_0\theta_1^{n_k}(0) \\ & \stackrel{?}{=} o(S_2^{gap,w}) \\ & = \omega^{\omega^\omega} \end{aligned}$$

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YES!

Sketch of proof

Is $\theta_0\theta_1\Omega_2 = \omega^{\omega^\omega}$?

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Proofsketch:

$$\begin{array}{rccc} \chi & : & \omega^{\omega^\omega} & \rightarrow & \theta_0\theta_1\Omega_2 \\ & & \omega^{\omega^{n-1}\cdot m} \cdot \alpha_m + \cdots + \omega^{\omega^{n-1}\cdot 0} \cdot \alpha_0 & \mapsto & \theta_0\theta_1^n\chi(\alpha_0)\ldots\theta_0\theta_1^n\chi(\alpha_m) \end{array}$$

$\alpha < \beta$ yields $\chi(\alpha) < \chi(\beta)$, by induction on $lh(\alpha) + lh(\beta)$.

Back to our question: gap-sequences with **more** than two labels

Are the theta-functions a maximal linear extension of the gap-sequences on three labels?
Meaning:

$$
\begin{aligned}
\text{Is} \quad & \sup_{n_1,\dots,q_l} \theta_0\theta_1^{n_1}\theta_2^{m_1}\dots\theta_1^{n_k}\theta_2^{m_k}\theta_0\dots\theta_0\theta_1^{p_1}\theta_2^{q_1}\dots\theta_1^{p_l}\theta_2^{q_l}(0) \\
& \stackrel{?}{=} o(\overline{S_3^{gap,w}}) \\
& = \omega^{\omega^{\omega^{\omega^{\omega}}}}
\end{aligned}
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Back to our question: gap-sequences with **more** than two labels

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NO!

Result

In general

Theorem

$\theta_0(\Omega_1) = \omega$,
$\theta_0\theta_1 \dots \theta_n\Omega_{n+1} = \omega_{n+2}$, *for* $n \geq 1$.

Sketch of proof

$$\theta_0\theta_1 \ldots \theta_n\Omega_{n+1} \leq \omega_{n+2}$$

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Definition

$f(\omega^{\alpha_1} + \alpha_2) := \omega^{\alpha_1} + f(\alpha_1) + f(\alpha_2)$

Sketch of proof

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Definition

$f(\omega^{\alpha_1} + \alpha_2) := \omega^{\alpha_1} + f(\alpha_1) + f(\alpha_2)$

- f is order preserving,
- $\omega^{\alpha_1} \leq f(\omega^{\alpha_1} + \alpha_2) < \omega^{\alpha_1+1}$,
- $f(\omega^{\alpha_1} + \alpha_2) = \omega^{\alpha_1} + f(\alpha_1) + f(\alpha_2)$ holds for non Cantor normal forms.

Idea of proof

Replace iteratively terms in θ_i (starting with the highest level) by terms in $\omega, +, \Omega_i$ in an order-preserving way such that terms of level 0 are below ε_0.

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Idea:
τ_i reduces $T_{\leq i} = \{t \in T\colon \text{only using } \theta_j \text{ with } j \leq i\}$ to $(T_{<i}, \omega, +)$.

Definition of τ_i

$$\begin{aligned} \tau_i 0 &:= 0, \\ \tau_i \theta_j \alpha &:= \theta_j \alpha \text{ if } j < i, \end{aligned}$$

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$$\tau_{i+1}\alpha := \Omega_{i+1}\alpha_1 + \omega^{f(\alpha_1)} K_i(\alpha) + \alpha_2,$$

with $\alpha_2 < \omega^{f(\alpha_1)}$ and $\omega^{f(\alpha_1)}, \alpha_1 < \varepsilon_0$.

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Define (if $i > 0$)

$$\tau_i \theta_i \alpha := \Omega_i \omega^{\alpha_1} + \omega^{\omega^{\alpha_1}} \left(\omega^{f(\alpha_1)} \cdot \tau_i K_i \alpha + \alpha_2 \right)$$

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One can prove that $\alpha < \beta$ implies $\tau_i \alpha < \tau_i \beta$.

Find collapsing functions $\overline{\theta}_0$, $\overline{\theta}_1$, $\overline{\theta}_2$, such that it captures well the maximal linear extension of the gap-sequences on three symbols, hence $\overline{\theta}_0\overline{\theta}_1\overline{\theta}_2\Omega_3 = \omega^{\omega^{\omega^{\omega^{\omega}}}}$.

$\rightarrow$ We have such collapsing functions $\overline{\theta}_i\alpha\beta$.

Thank you for your attention!

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