Finite iterations of infinite and finite Ramsey's theorem

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Outline

Independent statements from PA

It is well-known that several finite variations of Ramsey's theorem provide independent statements from Peano Arithmetic (PA).

- The first such example was found by Paris (in paper 1978). An "iteration versin of Finite Ramsey's theorem with relatively largeness".
- A simplification by Harrington (in manuscript 1977). "Paris-Harrington Principle: Finite Ramsey's theorem with relatively largeness".

Note that the original "iteration version" has the advantage:

it can approximate the infinite version of Ramsey's theorem.

Infinite vs finite Ramsey's theorem

Observation

Infinite Ramsey's theorem implies corresponding finite Ramsey's theorem (with some largeness notion).

However,

Fact

Infinite Ramsey's theorem as itself cannot prove the statement "for any m, m-th iteration of finite Ramsey's theorem holds".

This happens because of the lack of Σ^1_1-induction, but infinite Ramsey's theorem as itself does not prove such a strong induction.

What is needed for iterated Ramsey's theorem?

Question

What is a version of infinite Ramsey's theorem which implies iterated finite Ramsey's theorem?

$\Rightarrow$ We introduce several new variations of (infinite) Ramsey's theorem.
They are inhabited in rather strange places of, so-called, the Reverse Mathematics Zoo.

`http://rmzoo.uconn.edu/`

Outline

Ramsey's theorem

Ramsey's theorem is well-studied in reverse mathematics.

Definition (Ramsey's theorem.)

- RT^n_k: for any $P : [\mathbb{N}]^n \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $|P([H]^n)| = 1$.
- $\mathrm{RT}^n := \forall k\, \mathrm{RT}^n_k$. (In this talk, we may say RT^n_∞.)
- $\mathrm{RT} := \forall n\, \mathrm{RT}^n$. (In this talk, we may say $\mathrm{RT}^\infty_\infty$.)

Over RCA_0, we have the following:

- If $n' \leq n, k' \leq k$, then $\mathrm{RT}^n_k \Rightarrow \mathrm{RT}^{n'}_{k'}$.
- $\mathrm{RT}^n_k \Rightarrow \mathrm{RT}^n_{k+1}$.
- $\mathrm{RT}^{n+1}_2 \Rightarrow \mathrm{RT}^n$.

Thus, we have

$$\mathrm{RT}^1_2 \leq \mathrm{RT}^1 \leq \mathrm{RT}^2_2 \leq \mathrm{RT}^2 \leq \mathrm{RT}^3_2 \leq \mathrm{RT}^3 \leq \mathrm{RT}^4_2 \leq \dots$$

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Finite coloring

Definition (finite coloring)

- (n, k)-finite coloring is a function $P : [F]^n \to k$ where $F = \mathrm{dom}(P) \subseteq_{\mathrm{fin}} \mathbb{N}$.
- (n, ∞)-finite coloring is a function $P : [F]^n \to k$ where $F = \mathrm{dom}(P) \subseteq_{\mathrm{fin}} \mathbb{N}$ and $k \leq \min F$.
- (∞, ∞)-finite coloring is a function $P : [F]^n \to k$ where $F = \mathrm{dom}(P) \subseteq_{\mathrm{fin}} \mathbb{N}$ and $n, k \leq \min F$.

Density notion

Let $\alpha, \beta \in \omega \cup \{\infty\}$.

Definition (RCA_0)

- A finite set X is said to be 0-*dense*(α, β) if $|X| > \min X$.
- A finite set X is said to be $m+1$-*dense*(α, β) if for any (α, β)-finite coloring P with $\mathrm{dom}(P) = X$, there exists $Y \subseteq X$ which is m-dense(α, β) and P-homogeneous.

Note that "X is m-dense(α, β)" can be expressed by a Σ^0_0-formula.

Paris-Harrington principle

Definition

- $m\mathrm{PH}^{\alpha}_{\beta}$: for any $a \in \mathbb{N}$ there exists an m-dense(α,β) set X such that $\min X > a$.
- $m\widetilde{\mathrm{PH}}^{\alpha}_{\beta}$:for any $X_0 \subseteq_{\mathrm{inf}} \mathbb{N}$, there exists an m-dense(α,β) set X such that $X \subseteq_{\mathrm{fin}} X_0$.

We write $\mathrm{ItPH}^{\alpha}_{\beta}$ for $\forall m\, m\mathrm{PH}^{\alpha}_{\beta}$.

- Original Paris's independent statement from PA is ItPH^{3}_{2}.
- Original Paris-Harrington principle is $1\mathrm{PH}^{\infty}_{\infty}$.
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Paris's argument

We fix $\alpha, \beta \in \omega \cup \{\infty\}$ such that $\alpha, \beta \geq 2$, or $\alpha = 1$ and $\beta = \infty$.

Lemma

If (M, S) is a countable model of RCA_0 and $X \subset M$ ($X \in S$ and M-finite) is m-dense(α, β) for some $m \in M \setminus \omega$, then there exists a cut $I \subseteq_e M$ such that $I \cap X$ is unbounded in I and $(I, S \upharpoonright I) \models \mathrm{WKL}_0 + \mathrm{RT}^{\alpha}_{\beta}$. Here, $S \upharpoonright I = \{I \cap X \mid X \in S\}$.

This lemma means that m-dense(α, β) defines an indicator function for $\mathrm{WKL}_0 + \mathrm{RT}^{\alpha}_{\beta}$.

Paris's argument

Let $\tilde{\Pi}^0_3$ be a class of formulas of the form $\forall X \varphi(X)$ where $\varphi \in \Pi^0_3$.

Theorem (essentially due to Paris)

$\mathsf{WKL}_0 + \mathrm{RT}^\alpha_\beta$ *is a conservative extension of* $\mathsf{RCA}_0 + \{m\widetilde{\mathrm{PH}}^\alpha_\beta \mid m \in \omega\}$ *with respect to* $\tilde{\Pi}^0_3$*-sentences.*

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$\mathrm{ItPH}^\alpha_\beta$ *is not provable from* $\mathsf{WKL}_0 + \mathrm{RT}^\alpha_\beta$.

In fact, we can strengthen this result to the following.

Theorem

Over $\mathrm{I}\Sigma_1$, $\mathrm{ItPH}^\alpha_\beta$ *is equivalent to the* Σ_1*-soundness of* $\mathsf{WKL}_0 + \mathrm{RT}^\alpha_\beta$.

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Corollary

1. *The* $\tilde{\Pi}^0_3$*-part of* $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *is* $\mathrm{I}\Sigma^0_1 + \{m\widetilde{\mathrm{PH}}^2_2 \mid m \in \omega\}$.
2. *The* $\tilde{\Pi}^0_3$*-part of* $\mathsf{WKL}_0 + \mathrm{RT}^2_\infty$ *is* $\mathrm{I}\Sigma^0_1 + \{m\widetilde{\mathrm{PH}}^2_\infty \mid m \in \omega\}$.
3. $\mathrm{ItPH}^\infty_\infty$ *is not provable from* $\mathsf{ACA}_0 + \mathrm{RT}$.

Define GPH (generalized Paris-Harrington principle) as

> "every arithmetically definable infinite set contains m-dense(∞, ∞) set for any m".

Then, we have the following.

Theorem

$\mathrm{I}\Sigma_1 + \mathrm{GPH}$ *is the first-order part of* ACA_0'*, or equivalently* $\mathsf{ACA}_0 + \mathrm{RT}$.

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Outline

1. Introduction
2. Iteration of Finite Ramsey vs Infinite Ramsey
   - Finite coloring and density notion
   - Conservation and separation
3. A strengthened Ramsey's theorem
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   - Finite iteration
   - A stronger version of ACA_0'

Coloring family

Definition

A set $\mathcal{P}$ of (α, β)-finite coloring is said to be an (α, β)-coloring family if it is closed under subfunction, *i.e.*, if $P : [F]^n \to k \in \mathcal{P}$ and $H \subseteq F$, then, $P \upharpoonright [H]^n \in \mathcal{P}$.

- We write $X \in \mathrm{dom}(\mathcal{P})$ if for any $F \subseteq_{\mathrm{fin}} X$, there exists $P \in \mathcal{P}$ with $F = \mathrm{dom}(P)$.
- For an infinite $\bar{P} : [X]^n \to k$, we write $\bar{P} \in [\mathcal{P}]$ if for any $F \subseteq_{\mathrm{fin}} X$, $\bar{P} \upharpoonright [F]^n \in \mathcal{P}$.
- For $H \subseteq \mathbb{N}$ and $n, i \in \mathbb{N}$, define $\mathrm{Const}^n_{H,i}$ as $\mathrm{Const}(\bar{x}) = i$ for any $\bar{x} \in [H]^n$.
- An infinite set $H \subseteq \mathbb{N}$ is said to be homogeneous for $\mathcal{P}$ if $H \notin \mathrm{dom}(\mathcal{P})$ or $\mathrm{Const}^n_{H,i} \in \mathcal{P}$ for some i.

Strengthened Ramsey's theorem

Definition

For $\alpha, \beta \in \mathbb{N} \cup \{\infty\}$, $\mathrm{RT}^{\alpha+}_{\beta}$ is the following assertion:

> for any (α, β)-coloring family $\mathcal{P}$ and infinite $X \subseteq \mathbb{N}$, there exists an infinite set $H \subseteq X$ such that H is homogeneous for $\mathcal{P}$. then there exists an infinite homogeneous set for $\mathcal{P}$.

Proposition (RCA_0)

1. $\mathrm{RT}^{\alpha+}_{\beta} \Rightarrow \mathrm{RT}^{\alpha}_{\beta}$.
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Thus, $\mathrm{RT}^n_k, \mathrm{RT}^n_{<\infty}, \mathrm{RT}^{n+}_k, \mathrm{RT}^{n+}_{\infty}$ are all equivalent to ACA_0 for any standard $n \geq 3$ and $k \geq 2$.

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Ramsey type König's lemma $\mathrm{RKL}^{\alpha}_{\beta}$ is the following assertion:

> for any (α,β)-coloring family $\mathcal{P}$, if there exists an infinite set $X \in \mathrm{dom}(\mathcal{P})$, there exists an infinite function $\bar{P} \in \mathcal{P}$.

- RKL^1_2 is the original RKL introduced by Flood.
- WKL_0 implies $\mathrm{RKL}^{\alpha}_{\beta}$ for any α,β.

Proposition

$\mathrm{RT}^{\alpha+}_{\beta} \Leftrightarrow \mathrm{RT}^{\alpha}_{\beta} + \mathrm{RKL}^{\alpha}_{\beta}$.

Question

What is the strength of RKL^2_2?

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Finite iteration

Next, we consider finite iterations/simultaneous applications of Ramsey's theorem.

Definition

- $m\mathrm{RT}^n_k$: for any finite sequence $\langle P_i : [\mathbb{N}]^n \to k \mid i < m\rangle$, there exists an infinite set $H \subseteq \mathbb{N}$ such that H is homogeneous for any P_i.
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One can easily show by induction (outside of the system) that $\mathrm{RT}^{\alpha+}_\beta$ implies $m\mathrm{RT}^{\alpha+}_\beta$ for any $m \in \omega$ over RCA_0.

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Proposition (RCA_0)

1. ItRT^n_k *and* ItRT^n_∞ *are both equivalent to* RT^n.
2. ItRT *is equivalent to* RT.

Thus, it is not useful to think about the iteration of the usual Ramsey's theorem.

For a strengthened version, still we can easily see the following.

Proposition (RCA_0)

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Finite iteration of $\mathrm{RT}^{\alpha+}_{\beta}$

Theorem

Let $n \geq 3$ be a (standard) natural number. Then, ItRT^{n+}_{2} and $\mathrm{ItRT}^{n+}_{\infty}$ are both equivalent to RT over RCA_0.

Theorem (over RCA_0)

1. ItRT^{n+}_{2} *is strictly stronger than* RT^{n}_{2} *for any* $n \geq 2$.
2. $\mathrm{ItRT}^{n+}_{\infty}$ *is strictly stronger than* $\mathrm{RT}^{n}_{<\infty}$ *for any* $n \geq 1$.
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Finite iteration of $\mathrm{RT}^{\alpha+}_{\beta}$

Remark

1. $\mathrm{ItRT}^{\infty+}_{\infty}$ is provable from ACA_0^+.
   On the other hand, $(\omega, \mathrm{ARITH}) \models \mathrm{ItRT}^{\infty+}_{\infty}$.
   Thus, $\mathrm{ItRT}^{\infty+}_{\infty}$ does not imply ACA_0^+.
2. Similarly, ω-model of $\mathrm{RT}^2_2 + \mathrm{WKL}_0$ is a model of $\mathrm{ItRT}^{2+}_{\infty}$. Therefore, $\mathrm{ItRT}^{2+}_{\infty}$ does not imply ACA_0. However, we do not know whether ACA_0 implies $\mathrm{ItRT}^{2+}_{\infty}$ (or even ItRT^{2+}_{2}) or not.

Question

1. We have $\mathrm{RT}^1_{<\infty} + \mathrm{RT}^{1+}_2 \leq \mathrm{RT}^{1+}_{\infty} \leq \mathrm{ItRT}^{1+}_2 \leq \mathrm{ItRT}^{1+}_{\infty}$. Which inequalities are strict or not?
2. We have $\mathrm{RT}^2_{<\infty} + \mathrm{RT}^{2+}_2 \leq \mathrm{RT}^{2+}_{\infty} \leq \mathrm{ItRT}^{2+}_2 \leq \mathrm{ItRT}^{2+}_{\infty}$. Which inequalities are strict or not?

$\mathrm{ItRT}^{\alpha+}_{\beta}$ implies iterated Finite Ramsey

Theorem

Let $\alpha, \beta \in \omega \cup \{\infty\}$, $\alpha, \beta \geq 2$.
Then, RCA_0 proves the following.

$$\forall m (m\mathrm{RT}^{\alpha+}_{\beta} \Rightarrow m\widetilde{\mathrm{PH}}^{\alpha}_{\beta}).$$

Thus, over RCA_0, $\mathrm{ItRT}^{\alpha+}_{\beta}$ implies $\mathrm{It}\widetilde{\mathrm{PH}}^{\alpha}_{\beta}$, and particularly, it implies the Σ_1-soundness of $\mathrm{WKL}_0 + \mathrm{RT}^{\alpha}_{\beta}$.

In the above, we do not need $\mathrm{I}\Sigma^0_2$ in case $m \in \omega$.

Proof.

Let $m \in \mathbb{N}$ and X_0 be an infinite set. For $i < m+1$, define $\mathcal{P}_i$ as follows.

> $P : [F]^n \to k$ is a member of $\mathcal{P}_i$ if and only if $F \subseteq_{\mathrm{fin}} X_0$, $n \leq \min\{\alpha, \min X\}$, $k \leq \min\{\beta, \min X\}$ and any P-homogeneous set $Y \subseteq F$ is not $(m-i-1)$-dence(α,β).

(By the definition, $\mathrm{dom}(\mathcal{P}_0) \supseteq \mathrm{dom}(\mathcal{P}_1) \supseteq \ldots$.)

Let $\bar{m}$ be the least $i < m+1$ such that $\mathrm{dom}(\mathcal{P}_i)$ does not contain X_0. (Trivially, $\mathrm{dom}(\mathcal{P}_m)$ does not contain any infinite set.)

If $\bar{m} = 0$, we have done.
Assume $\bar{m} > 0$, then we can apply $\bar{m}\mathrm{RT}^{\alpha+}_{\beta}$ and $\langle \mathcal{P}_i \mid i < \bar{m} \rangle$, and let $H \subseteq X_0$ be an infinite common homogeneous set. Then, any finite subset of H is $(m-\bar{m}-1)$-dence(α,β), and thus $H \in \mathrm{dom}(\mathcal{P}_{\bar{m}})$, which is a contradiction.

□

Corollary

1. $\mathsf{WKL}_0 + \mathrm{RT}^{2+}_2$, $\mathsf{RCA}_0 + \mathrm{RT}^{2+}_2$ *and* $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *have the same* $\tilde{\Pi}^0_3$*-part, namely* $\mathrm{I}\Sigma^0_1 + \{m\widetilde{\mathrm{PH}}^2_2 \mid m \in \omega\}$.
2. $\mathsf{WKL}_0 + \mathrm{RT}^{2+}_\infty$, $\mathsf{RCA}_0 + \mathrm{RT}^{2+}_\infty$ *and* $\mathsf{WKL}_0 + \mathrm{RT}^2_{<\infty}$ *have the same* $\tilde{\Pi}^0_3$*-part, namely* $\mathrm{I}\Sigma^0_1 + \{m\widetilde{\mathrm{PH}}^2_\infty \mid m \in \omega\}$.
3. ItRT^{2+}_2 *implies the* Σ_1*-soundness of* $\mathsf{WKL}_0 + \mathrm{RT}^2_2$.
4. $\mathrm{ItRT}^{2+}_\infty$ *implies the* Σ_1*-soundness of* $\mathsf{WKL}_0 + \mathrm{RT}^2_\infty$.
5. ItRT^{3+}_2 *implies the* Σ_1*-soundness of* ACA_0.
6. $\mathrm{ItRT}^{\infty+}_\infty$ *implies the* Σ_1*-soundness of* $\mathsf{ACA}_0 + \mathrm{RT}$*, or equivalently* ACA_0'.

A stronger version of ACA_0'

Theorem

The following are equivalent over RCA_0.

1. ACA_0'': *for any sequence of Turing functionals* $\langle \Phi_{e_i} \mid i < m \rangle$ *, and for any* Z*, there exists a sequence* $\langle Z^{(k_i)} \mid i \leq m \rangle$ *such that* $k_0 = 0$ *and* $k_{i+1} = k_i + \Phi_{e_i}^{Z^{(k_i+1)}}(0)$.
2. $\mathrm{ItRT}_{\infty}^{\infty+}$.

Questions

Question

1. Is RT^2_2 equivalent to RT^{2+}_2 over RCA_0?
2. Is RT^2_∞ equivalent to RT^{2+}_∞ over RCA_0?

Question

What is the strength of RKL^2_2?

Question

Is there a “simpler” version of ItRT^+?

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