On the “finitary” infinite Ramsey's theorem

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Overview:

1. Motivation
2. “finitary” Ramsey
3. Inserting the parameter in “finitary” Ramsey
4. Some logical strengths for different values

Motivation

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From the study of first order concrete mathematical incompleteness there is the phenomenon of the *phase transition*:

Given some $T \nvdash \varphi$, examine the parametrised version φ_f.

Classify parameter values f according to the provability of φ_f.

Results in this programme follow certain heuristics.

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From the study of first order concrete mathematical incompleteness there is the phenomenon of the *phase transition*:

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Question: Do we have something similar in Reverse Mathematics?

Remarks

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Theorems examined in reverse mathematics have no obvious parametrisation.

The “finitary” infinite Ramsey’s theorem

“finitary” Ramsey

The “finitary” pigeonhole principle was introduced by Tao, examined by Gaspar and Kohlenbach.

We examine the generalisation to Ramsey’s theorem.

“finitary” Ramsey

Definition (AS)

$F\colon \{(\text{codes of) finite sets}\} \to \mathbb{N}$ is asymptotically stable if for every sequence $X_1, X_2, \ldots$ of finite sets there is i such that $F(X_i) = F(X_j)$ for all $j > i$.

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Definition

1. $[X]^d =$ set of d-element subsets of X
2. $[a, b]^d = [\{a, \ldots, b\}]^d$
3. For $C\colon [X]^d \to [0, c]$ a set $H \subseteq X$ is C-homogeneous if C restricted to $[H]^d$ is constant.

“finitary” Ramsey

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For every $C\colon \mathbb{N}^d \to [0,k]$ there exists infinite C-homogeneous $H \subseteq \mathbb{N}$.

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For every $F \in \mathrm{AS}$ there exists R such that for all

$$C \colon [0, R]^d \to [0, k]$$

there exists C-homogeneous H of size $> F(H)$.

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Theorem

$\mathrm{WKL}_0 \vdash \mathrm{FRT}^k_d \leftrightarrow \mathrm{RT}^k_d$.

Inserting the parameter

Inserting the parameter:

We can use subsets of AS as parameter values:

Definition ($\mathrm{FRT}_d^k(G)$)

For every $F \in G$ there exists R such that for all

$$C\colon [0, R]^d \to [0, k]$$

there exists C-homogeneous H of size $> F(H)$.

Some values:

Definition (CF)

Constant functions

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$$\{F \in \mathrm{AS} : \exists m \forall X. F(X) \leq \max\{\min X, m\}\}$$

Definition (MD)

$$\{F \in \mathrm{AS} : \min X = \min Y \to F(X) = F(Y)\}$$

Some logical strengths for different values

Strengths for values

$\mathrm{FRT}(\mathrm{CF})$ is the finite Ramsey's theorem, which is known to be provable in RCA_0.

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$\mathrm{FRT}(\mathrm{CF})$ is the finite Ramsey's theorem, which is known to be provable in RCA_0.

$\mathrm{FRT}(\mathrm{UI})$ is equivalent to the Paris–Harrington principle, which is known to be equivalent to 1-$\mathrm{Con}(\mathrm{I}\Sigma_d)$ when the dimension is fixed to $d+1$.

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Definition

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Theorem

$\mathrm{RCA}_0 \vdash \mathrm{FRT}_d(\mathrm{MD}) \leftrightarrow \mathrm{WO}(\omega_d)$

Summary

RCA_0 proves the following:

$$
\begin{array}{lcllcl}
\mathrm{FRT} & \leftrightarrow & \mathrm{RT} & & & \\
\mathrm{FRT}^k_d & \leftarrow & \mathrm{RT}^k_d \text{ for } (d > 2) & & & \\
\mathrm{FRT}^k_d & \rightarrow & \mathrm{RT}^k_d & & & \\
\mathrm{FRT(MD)} & \leftrightarrow & \mathrm{AR} & & \leftrightarrow & \mathrm{WO}(\varepsilon_0) \\
\mathrm{FRT}_{d+1}\mathrm{(MD)} & \leftrightarrow & \mathrm{AR}_d & & \leftrightarrow & \mathrm{WO}(\omega_{d+1}) \\
\mathrm{FRT(UI)} & \leftrightarrow & \text{1-consistency of } \mathrm{PA} & & & \\
\mathrm{FRT}_{d+1}\mathrm{(UI)} & \leftrightarrow & \text{1-consistency of } \mathrm{I}\Sigma_d & & & \\
\mathrm{FRT(CF)} & & & & &
\end{array}
$$

Furthermore, $\mathrm{WKL}_0 \vdash \mathrm{RT}^k_d \rightarrow \mathrm{FRT}^k_d$.

Thank you for listening.

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