Some Progress on Kierstead's Conjecture

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Computable Linear Orderings

$(L, \leq)$ is a computable linear ordering, if $\leq$ is a linear ordering on L and both L and $\leq$ are computable.

Some order types:

- ω;
- ω^*;
- η;
- ζ;
- addition and product.

Folklore

There is a computable linear ordering L of order type ω with $S(x)$, the successor function, not computable.

References:

- Computability Theory and Linear Orderings, Rod Downey, Chapter 14 in "Handbook of Recursive Mathematics".
- Linear Orderings, J. G. Rosenstein, 1982.

About η

Folklore
η is computably categorical (or autostable).

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Remmel's Characterization

A computable linear ordering $(L, \leq)$ is computably categorical if and only if it has only finitely many successivities.

Effective considerations: Suborderings

A classical result

Any infinite linear ordering has an infinite subordering of order type either ω or ω^*.

Theorem (Tennenbaum, Denisov)

There is a computable linear ordering of order type $\omega + \omega^*$ with no infinite computably enumerable suborderings of order type ω or ω^*.

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View from reverse math.

More on Effective considerations

Theorem (Rosenstein)

If $(L, \leq)$ is a computable linear ordering, then it has a computable subordering of type $\omega, \omega^*, \omega + \omega^*$ or $\omega + \zeta\eta + \omega^*$.

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Theorem (Lerman)

There is a computable linear ordering with no computable subordering of type ω, ω^*, or $\omega + \omega^*$.

Theorem (Manaster)

If $(L, \leq)$ is an infinite computable linear ordering, then L has a Π_1 subset of type ω or ω^*.

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There is a computable linear ordering $(L, \leq)$ such that if f is a nontrivial self-embedding of L, then f computes $\emptyset'$.

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Theorem (Downey, Jockusch and Miller)
There is a computable linear ordering of order type ω with no nontrivial $\mathbf{0}'$-computable self-embedding.

Computable rigidity

A linear ordering is computably rigid if it has no nontrivial computable automorphisms.

Theorem (Schwartz)

A computable linear ordering has a computably rigid copy if and only if it has no interval of order type η.

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Theorem (Kierstead)

There is a computable linear ordering of type $2 \cdot \eta$ with no nontrivial Π_1 automorphism.

Kierstead's conjecture

Definition (Kierstead)

An automorphism is fairly trivial if for all x, $[x, f(x)]$ is finite.
An automorphism is strongly nontrivial a nontrivial automorphism is not fairly trivial.

Kierstead's Conjecture

For a computable linear ordering $\mathcal{L}$, every computable copy of $\mathcal{L}$ has a strongly nontrivial Π_1 automorphism if and only if the corresponding order type contains an interval of order type η.

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Downey and Moses proved that it is also true for discrete computable linear orderings.

Here a linear ordering is discrete if every element has both an immediate predecessor and an immediate successor, except for the possible first and last elements.

η-like

A linear ordering $\mathcal{L}$ is η-like if $\mathcal{L}$ is isomorphic to

$$\sum_{q \in \mathbb{Q}} F(q),$$

where F is a function from $\mathbb{Q}$ to $\mathbb{N} \setminus \{0\}$.

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Theorem (Harris, Lee and Cooper)

Suppose that $F : \mathbb{Q} \to \ \mathbb{N} \setminus \{0\}$ is $\emptyset'$-limitwise monotonic and that the linear ordering $\mathcal{L} \simeq \sum_{q \in \mathbb{Q}} F(q)$ has no dense intervals. Then $\mathcal{L}$ has a computable copy with no (strongly) nontrivial Π_1-automorphisms.

- This theorem improves Kierstead's result a lot.

Extended $\emptyset'$-limitwise monotonic function

A function $F : \mathbb{Q} \to (\mathbb{N} \setminus \{0\}) \cup \{\zeta\}$ is an extended $\emptyset'$-limitwise monotonic function if we assume $\zeta > n$ for each $n \in \mathbb{N}$ and there is a $\mathbf{0}'$-limitwise monotonic function $f : \mathbb{Q} \times \mathbb{N} \to (\mathbb{N} \setminus \{0\}) \cup \{\zeta\}$ such that

1. for all $q \in \mathbb{Q}, s \in \mathbb{N}$, $f(q,s) \leq f(q,s+1)$;
2. for all $q \in \mathbb{Q}$, $\lim\limits_{s \to \infty} f(q,s) = F(q)$;
3. if $\lim\limits_{s \to \infty} f(q,s) = \zeta$, then there is an s_0 such that for all $s \geq s_0$, $f(q,s) = \zeta$.

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For an extended $\emptyset'$-limitwise monotonic function F, we define linear ordering $\sum_{q\in\mathbb{Q}} F(q)$.

- This notion extends the one considered by Harris, Lee and Cooper, and maybe by Turetsky and Kach.
- $2\cdot\eta + \zeta + 3\cdot\eta$, $\zeta\cdot\eta$ are in our consideration, but not $\zeta\cdot\omega$.

Almost trivial automorphisms

An automorphism f of a linear ordering $\mathcal{L} = (L, \leq)$ is almost trivial if

$$(\forall x)\big[|[x]_{\mathcal{L}}| > 1 \quad \rightarrow \quad f([x]_{\mathcal{L}}) = [x]_{\mathcal{L}}\big].$$

- For discrete linear orderings, there is no difference between "fairly trivial" and "almost trivial".

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Theorem (Wu and Zubkov)

Suppose that F is an extended $\emptyset'$-limitwise monotonic function and that the linear ordering $\mathcal{L} \simeq \sum_{q \in \mathbb{Q}} F(q)$ has no dense intervals. Then $\mathcal{L}$ has a computable copy with only almost trivial Π_1-automorphisms.

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- This generalizes Harris-Lee-Cooper's result, and covers some instances of Downey-Moses' result.

Thanks!