On the interpretation of HPC in the Kreisel-Goodman Theory of Constructions

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Background project (joint with Walter Dean)

Goal: to rehabilitate a form of construction-based semantics for inutitionistic logic and mathematics known as the **Theory of Constructions** [ToC] originally proposed by Kreisel (1962) and Goodman (1968).

1) Provide a construction-based system in which Heyting Predicate Calculus [HPC] can be interpreted ($\mathcal{T}_0$).
2) Extend $\mathcal{T}_0$ to $\mathcal{T}_1$ in which Heyting Arithmetic [HA] can be interpreted.
3) Extend this to HA^ω and reprove **Goodman's Theorem**:

$$\mathrm{HA}^\omega + \mathrm{AC} \text{ is conservative over } \mathrm{HA}.$$

4) Proceed in a manner which is compatible with several adequacy conditions on the proper analysis of "**constructive validity**".

This is a report of this on-going project.

Kreisel's program

Our main purpose here is to enlarge the stock of formal rules of proof which follow directly from the meaning of the basic intuitionistic notions but not from the principles of classical mathematics so far formulated. The specific problem which we have chosen to lead us to these rules is also of independent interest: *to set up a formal system, called [the] 'abstract theory of constructions' for the basic notions mentioned above, in terms of which formal rules of Heyting's predicate calculus can be interpreted.*

In other words, we give a formal semantic foundation for intuitionistic formal systems in terms of the abstract theory of constructions. This is analogous to the semantic foundation for classical systems Tarski (1935) in terms of abstract set theory.

Kreisel (1962) "Foundations of intuitionistic logic"

Outline

I) The intended (i.e. BHK) interpretation and the analysis of "constructive validity".

II) The Kreisel-Goodman programme:

- language: the proof predicate πst
- rules about proofs: decidability, internalization, reflection
- the Kreisel-Goodman paradox
- responding to the paradox

III) Rehabilitating the programme:

- interpreting HPC: impredicativity & the second clause
- soundness

Intuitionistic implication in BHK

The implication $A \to B$ can be asserted, if and only if we possess a construction $\mathfrak{r}$, which, joined to any construction proving A (presuming that the latter be effected), would automatically effect a construction proving B. Heyting (1956)

Naive observations:

1) The original formulations **do** treat constructions as “first class objects” (e.g. by quantifying over them).
2) But they **do not** (at least *explicitly*) mention type distinctions.

The Troelstra & van Dalen [1988] formulation of BHK

($P_\bot$) $\bot$ has no proof.

($P_\land$) A proof of $A \land B$ consists of a proof of A and a proof of B.

($P_\lor$) A proof of $A \lor B$ consists of a proof of A or a proof of B.

($P_\to$) A proof of $A \to B$ consists of a construction which transforms any proof of A into a proof of B.*

($P_\neg$) A proof of $\neg A$ consists of a construction which transforms any hypothetical proof of A into a proof of $\bot$.*

($P_\forall$) A proof of $\forall x A$ consists of a construction which transforms all c in the intended range of quantification into a proof of $A(c)$.*

($P_\exists$) A proof of $\exists x A$ consists of an object c in the intended range of quantification together with a proof of $A(c)$.

* In (e.g.) Troelstra (1977) and van Dalen (1973), "K" stands for "Kreisel" and there are **second clauses** for $\to$, $\neg$, and $\forall$.

Construction-based semantics

The reason that A is *intuitionistically* (*constructively*, if you prefer) *valid* is that there is a specific term t such that $\vdash t \in A$ is provable in the theory of constructions. Scott (1970) "Constructive Validity"

- **Goal**: treating constructions $s, t, u, \ldots$ as *primitives*, analyze the BHK clauses so as to prove

  $$\mathrm{HPC} \vdash A \text{ if and only if } \vdash \mathrm{Pr}(A, t)$$

  for some t which is a formal term of the theory and

  $\mathrm{Pr}(A, t)$ formalizes $\underline{t \textit{ satisfies the BHK proof conditions of } A}$.

- Goodman (1970)'s goal: provide a "**type- and logic-free**" foundation for intuitionistic logic and mathematics.

Brouwer-Heyting-**Kreisel** interpretation

The Kreisel (1962) proposal:

($K_\wedge$) $\Pi(A \wedge B, s) := \lambda\vec{x}.(\Pi(A, D_1 s) \cap_k \Pi(B, D_2 s))$

($K_\rightarrow$) $\Pi(A \rightarrow B, s) := \pi(\lambda y.(\Pi(A, y) \supset_k \Pi(B, (D_2 s)y)), D_1 s)$

Compare: $\Pi(A \rightarrow B, s) := \lambda\vec{x}.\lambda y.(\Pi(A, y) \supset_k \Pi(B, sy))$

The clause ($K_\rightarrow$) formalizes

$s = \langle s_1, s_2 \rangle$ *is a proof* $A \rightarrow B$ *just in case* s_1 *is a proof that for all* y*, if* $\Pi(A, y)$*, then* $\Pi(B, s_2 y)$

- The requirement on s_1 is the "**second clause**" added to ensure the decidability of $K_\rightarrow$, $K_\neg$, and $K_\forall$.
- Why worry about **decidability**? To ensure that the proof conditions of $\rightarrow$, $\neg$, $\forall$ do not quantify over "all proofs" in a impredicative/circular manner.

Gödel on BHK

> [The Heyting interpretation does] violate the principle ... that the word "any" can be applied only to those totalities for which we have a finite procedure for generating all their elements. For the totality of all possible proofs certainly does not possess this character, and nevertheless the word "any" is applied to this totality in Heyting's axioms, as you can see from the example which I mentioned before, which reads: "Given **any** proof for a proposition p, you can construct a reductio ad absurdum for the proposition $\neg p$". Totalities whose elements cannot be generated by a well-defined procedure are in some sense vague and indefinite as to their borders. And this objection applied particularly to the totality of intuitionistic proofs because of the vagueness of the notion of constructivity.

Gödel (1933) "The present situation in the foundations of mathematics"

Red herrings?

Goodman (1968)'s main result is "Goodman's theorem":

$$\mathrm{HA}^{\omega} + \mathrm{AC} \text{ is conservative over } \mathrm{HA}^{\omega} \text{ (and hence } \mathrm{HA}\text{)}.$$

But ToC has been dismissed for several reasons – e.g.:

1) a paradox (Kreisel-Goodman paradox)
   - due to "reflection principle" (provability implies truth).
   - due to "the second clause" ? (Weinstein (1983). But we claim "No.")
2) Goodman's unmotivated solutions (stratification of the domain of proofs)

However:

- For some theoretical purposes, "reflection" is not used.
- Then no worries about the Paradox and stratification
- The second clause makes some sense to make sure "decidability"

The theory $\mathcal{T}^*$: syntax

We will first present an **inconsistent** theory $\mathcal{T}^*$ similar to that of Kreisel (1962) before isolating a **consistent** subtheory $\mathcal{T}_0 \subseteq \mathcal{T}^*$.

- Terms:
  $s := x, y, z \ldots \mid \top \mid \bot \mid c_{\mathfrak{p}} \mid Dtu \mid D_1 t \mid D_2 t \mid \lambda x.t \mid tu \mid \pi ut$
  $\top, \bot$ (truth values), D (pairing), D_1, D_2 (projection)
- Following Curry and Feys (1958), Goodman took D, D_1, D_2 as primitives. But we don't have to:
  $D =_{df} \lambda x.\lambda y.\lambda z.zxy$, $D_1 =_{df} \lambda p.p\top$, $\quad D_2 =_{df} \lambda p.p\bot$
- Formulas: $s \equiv t$
- Since $\mathcal{T}^*$ is based on the **untyped lambda calculus**, terms need not always be defined (i.e. reduce to a normal form).

The theory $\mathcal{T}^*$: axioms, sequents, rules

- $\mathcal{T}^*$ is a single conclusion sequent calculus consisting of
  1) structural rules: weakening, substitution
  2) the equational theory of $\lambda\beta\eta$-equality (cf. [Hindley, 1986])
  3) special axioms and rules about π
- Sequents: $\Delta \vdash s \equiv t$
- Intended interpretation:

$$\pi st \equiv \top \text{ iff } t \text{ is a constructive proof of } s \equiv \top$$

- Special rules about π:

$$(\textsc{Dec}) \ \frac{\Delta, \pi uv \equiv \bot \vdash s \equiv t \quad \Delta, \pi uv \equiv \top \vdash s \equiv t}{\Delta \vdash s \equiv t}$$

$$(\textsc{ExpRfn}) \ \Delta, \pi st \equiv \top \vdash s \equiv \top$$

$$(\textsc{Int}) \ \vdash s \equiv \top \text{ with derivation } \mathfrak{p}, \text{ then } \vdash \pi s c_{\mathfrak{p}} \equiv \top$$

Motivating the rules: $\textsc{Dec}$

$$\frac{\Delta, \pi uv \equiv \bot \vdash s \equiv t \quad \Delta, \pi uv \equiv \top \vdash s \equiv t}{\Delta \vdash s \equiv t}$$

- $\textsc{Dec}$ is intended to formalize that $\pi st \equiv \top$ is **decidable**.
- Reconstructing Kreisel's motivation:
  - Kreisel (1965): "we recognize a proof when we see one"
  - Analogy with $\mathrm{T} \vdash \mathtt{Proof}_{\mathrm{T}}(\mathtt{n}, \ulcorner A \urcorner)$ or $\mathrm{T} \vdash \neg\mathtt{Proof}_{\mathrm{T}}(\mathtt{n}, \ulcorner A \urcorner)$ since $\mathtt{Proof}_{\mathrm{T}}(x, y)$ is a Δ^0_1-formula.
  - The goal is to define $\Pi(A, s)$ in terms of π so that that it too is decidable in the sense of $\textsc{Dec}$.
- NB: what $\textsc{Dec}$ really formalizes is that πst may be assumed to be **always defined** and equal to $\top$ or $\bot$.

Motivating the rules: Int

$$\vdash s \equiv \top \text{ with derivation } \mathfrak{p}, \text{ then } \vdash \pi s c_{\mathfrak{p}} \equiv \top$$

- Int is a form of *internalization principle* – i.e. if s is *provable*, then we can "internalize" its derivation within $\mathcal{T}^*$.
- Arithmetical analogue:

  (HB1) If $\vdash A$, then exists $n \in \mathbb{N}$ s.t. $\vdash \mathtt{Proof}_{\mathrm{T}}(\mathtt{n}, \ulcorner A \urcorner)$

- Status:
  - This is an again intuitively plausible principle about constructive provability.
  - But the reason K & G included it appears to have been *intrustrumental* – i.e. it's needed to secure the decidability of the proof condition for $A \to B$.

On the rule ExpRfn

$$\pi st \equiv \top \vdash s \equiv \top$$

- ExpRfn is a form of *reflection principle* – i.e. if s is *proven* by t, then s is true.
- Kreisel (1962): "intuitively obvious on the intended interpretation"
- Arithmetical analogy

$$\mathrm{Rfn(T)} \qquad \texttt{Proof}_{\mathrm{T}}(x, \ulcorner A \urcorner) \to A$$

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$$\mathrm{Rfn}(\mathrm{T}) \qquad \texttt{Proof}_{\mathrm{T}}(x, \ulcorner A \urcorner) \to A$$

- ExpRfn also isn't needed to prove the soundness of Goodman's interpretation of HPC or HA.
- Note that ExpRfn is formulated in the *interpreting* theory, and there is no way of expressing reflection via Π.

Background for the Kreisel-Goodman paradox

- Goodman (1970) sketched a derivation of a contradiction in $\mathcal{T}^*$ which resembles Montague (1963)'s *paradox*:
  - self-reference (e.g. $\mathrm{T} \supseteq \mathrm{Q}$ satisfies the Diagonal Lemma)
  - a "provability like" predicate $P(x)$

$$\begin{array}{lll} (\mathrm{Rfn}) & P(\ulcorner A \urcorner) \rightarrow A & \text{(reflection)} \\ (\textsc{Nec}) & \vdash A \ \therefore \vdash P(\ulcorner A \urcorner) & \text{(necessitation)} \end{array}$$

- In $\mathcal{T}^*$ we get self-reference via *fixed-point combinators* – e.g.

$$Y =_{\mathsf{df}} \lambda t.(\lambda x.t(xx))(\lambda x.t(xx))$$

is s.t. $\vdash Yt \equiv t(Yt)$.

- A term which is "equivalent to is own unprovability"
  - Let $h(y,x) =_{\mathsf{df}} \lambda y.\lambda x.(\pi yx \supset_1 \perp)$.
  - Then $\vdash Y(h(y,x)) \equiv h(Y(h(y,x)),x)$.

Derivations

Montague (1963)	$\approx$ Goodman (1970)	
$\vdash D \leftrightarrow \neg P(\ulcorner D \urcorner)$	$\vdash Y(h(y,x)) \equiv h(Y(h(y,x)),x)$	FP
$\vdash P(\ulcorner D \urcorner) \to D$	$\pi(Y(h(y,x)))x \equiv \top \vdash Y(h(y,x)) \equiv \top$	ExpRfn
	$\pi(Y(h(y,x)))x \equiv \top \vdash h(Y(h(y,x)),x) \equiv \top$	
	$\pi(Y(h(y,x)))x \equiv \top \vdash (\pi(Y(h(y,x)))x \supset_1 \bot) \equiv \top$	
	$\pi(Y(h(y,x)))x \equiv \top \vdash \bot \equiv \top$	
$\vdash \neg P(\ulcorner D \urcorner)$	$\vdash \pi(Y(h(y,x)))x \equiv \bot$	Dec
	$\vdash (\pi(Y(h(y,x)))x \supset_1 \bot) \equiv \top$	
	$\vdash h(Y(h(y,x)),x) \equiv \top$	
$\vdash D$	$\vdash Y(h(y,x)) \equiv \top$	
$\vdash P(\ulcorner D \urcorner)$	$\vdash \pi(Y(h(y,x)))c_{\mathfrak{p}} \equiv \top$	Int
	$\vdash \pi(Y(h(y,x)))c_{\mathfrak{p}} \equiv \bot$	subst
$\vdash \bot$	$\vdash \top \equiv \bot$	

The theory $\mathcal{T}_0$

Out view: the paradox has nothing to do with "the second clause."

- Goal: to isolate a sub-theory $\mathcal{T}_0$ of $\mathcal{T}^*$ s.t.
  - consistent
  - unstratified (and hence *more* "type- and logic-free")
  - can be used to interpret the BHK clauses such that soundness and completeness of HPC are provable
- **Proposal**: Reflection isn't used in Goodman's original interpretation of HPC in $\mathcal{T}^*$. So we consider the system $\mathcal{T}_0 = \mathcal{T}^* - \textsc{ExpRfnR}$.
- Other options are available:
  - Prohibit the application Int to consequences of $\textsc{ExpRfn}$.
  - A finer grained treatment of internalization resembling "**lifting**" in the sense of Artemov (2001)'s LP.

The mapping $\Pi(A, x)$

We want to define a mapping

$$\Pi : \mathrm{Form}_{\mathrm{HPC}} \times \mathrm{Term}_{\mathcal{T}_0} \to \mathrm{Term}_{\mathcal{T}_0}$$

s.t. $\Pi(A, s) \equiv \top$ expresses "***s* is a proof of *A***" à la BHK and is a decidable predicate.

- Straightforward in the case of $\wedge$, $\vee$, $\exists$.
- Trickier in the case of $\rightarrow$ in virtue of the putative impredicativity of $(\mathsf{P}_{\rightarrow})$ – e.g.

  $\mathrm{Pr}(A \rightarrow B, t)$ iff for **all** proof s, if $\mathrm{Pr}(A, s)$, then $\mathrm{Pr}(B, t(s))$

- This seems analogous to a Π^0_1 statement about $\mathbb{N}$ – i.e *a priori* undecidable.

Interpreting the propositional calculus in $\mathcal{T}_0$

- The Church interpretation of the classical connectives in the untyped lambda calculus:

$$
\begin{array}{ll}
\top = \lambda x.\lambda y.x & \bot = \lambda x.\lambda y.y \\
\cap = \lambda a.\lambda b.a(b\bot) & \cup = \lambda a.\lambda b.a(\top b) \\
\supset = \lambda a.\lambda b.a(b\top) & \sim = \lambda a.a(\bot\top)
\end{array}
$$

- E.g. $\vdash \bot \cup \top \equiv (\lambda a.\lambda b.a(\top b))(\lambda x.\lambda y.y)(\lambda x.\lambda y.x) \equiv (\lambda b.((\lambda x.\lambda y.y)(\top b))((\lambda x.\lambda y.x) \equiv (\lambda x.\lambda y.y)(\top(\lambda x.\lambda y.x)) \equiv \lambda x.\lambda y.x =_{df} \top$

The Brouwer-Heyting-**Kreisel** interpretation

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($K_\rightarrow$) $\Pi(A \rightarrow B, s) := \pi(\lambda y.(\Pi(A, y) \supset_k \Pi(B, (D_2 s)y)), D_1 s)$

($K_\neg$) $\Pi(\neg A, s) := \lambda\vec{x}.\pi(\lambda y.(\Pi(A, y) \supset_k \Pi(\bot, (D_2 s)y)), D_1 s)$

($K_\forall$) $\Pi(\forall z A(z), s) := \lambda\vec{x}.\pi(\lambda y.\Pi(A[y/z], (D_2 s)y), D_1 s)$

($K_\exists$) $\Pi(\exists z A(z), s) := \lambda\vec{x}.\Pi(A[(D_2 s)/z], D_1 s)$

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The requirement on s_1 is the "**second clause**" added to ensure the decidability of K$_\rightarrow$, K$_\neg$, and K$_\forall$.

Formulating soundness and completeness

K & G both state versions of the following (K w/o proof):

Theorem $\mathrm{HPC} \vdash A$ iff there is a term t s.t. $\mathcal{T} \vdash \Pi(A, t) \equiv \top$.

- Our claim: The RHS gives an analysis of constructive validity using ToC à la Scott (1970).
- The L-to-R direction expresses a form of *soundness* – i.e.

$$\text{if } \mathrm{HPC} \vdash A, \text{ then } \models_i A.$$

- The R-to-L direction expresses a form of *completeness* – i.e.

$$\text{if } \models_i A, \text{ then } \mathrm{HPC} \vdash A.$$

- Our goal is to prove these results for $\mathcal{T} = \mathcal{T}_0$.

Soundness (1)

- We show **by induction on** HPC **derivations** that

$$\mathrm{HPC} \vdash A \quad \Rightarrow \quad \mathcal{T}_0 \vdash \Pi(A, t)$$

- E.g. some axioms:
  - For $A \to A$, we may take $t = Dc_{\mathfrak{p}}(\lambda x.x)$ where $\mathfrak{p}$ is a proof of $(\Pi(A, x) \supset \Pi(A, x)) \equiv \top$.
  - For $(A \wedge B) \to A$, we may take $t = Dc_{\mathfrak{p}}(\lambda x.D_1 x)$
- For *modus ponens* we have

$$\text{If } \vdash \Pi(A \to B, s) \equiv \top \text{ and } \vdash \Pi(A, t) \equiv \top,$$
$$\text{then } \vdash \Pi(B, D_2 st) \equiv \top.$$

- So just like Curry-Howard:
  - *modus ponens* $\sim$ β-conversion
  - deduction theorem $\sim$ λ-abstraction

Soundness (2): An observation on a "deduction theorem."

Suppose that x does not occur in Δ and

$$\Delta, \Pi(A, x) \equiv \top \vdash \Pi(B, s) \equiv \top$$

Then $\Delta \vdash \Pi(A \to B, t)$ for some term t.

Proof: The hypotheses imply $\Delta, \Pi(A, x) \equiv \top \vdash \Pi(B, \lambda x.s'x) \equiv \top$ (for some s') and thus since $\Pi(A, x)$ is **decidable**, we have by the truth functional Deduction Theorem that

$$\Delta \vdash (\Pi(A, x) \supset \Pi(B, \lambda x.s'x)) \equiv \top$$

By applying the rule $\textsc{Int}$ there is hence a term $c_{\mathfrak{p}}$ such that

$$\Delta \vdash \pi((\Pi(A, x) \supset \Pi(B, \lambda x.s'x)), c_{\mathfrak{p}}) \equiv \top$$

So we may take $t = Dc_{\mathfrak{p}}(\lambda x.s')$ as the term s.t. $\Delta \vdash \Pi(A \to B, t) \equiv \top$.

Consistency

Before proving the soundness of the "Kreisel interpretation" of HPC, we must show that the revised system $\mathcal{T}_0$ is **consistent**.
Outline of a consistency proof of ToC, i.e. $\mathcal{T}_0$, à la Goodman (1968).

1) Define a "deterministic" reduction relation for terms of ToC
2) Define a notion of satisfaction and validity.
   - $t \equiv s$ is satisfiable with respect to a sequence giving a substitution of a free variable with reduced terms if t and s reduced to the same term.
   - $t \equiv s$ is valid if it is satisfiable with respect to every such sequence.
3) Show that all sequents $\Delta \vdash t \equiv s$ derivable in $\mathcal{T}_0$ are valid.
4) Observe that (e.g.) $\lambda xy.x \equiv \lambda xyz.xz(yz)$ is not valid.

We have yet to fill out some "gaps" that Goodman left in his text.

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