Embeddability amongst the countable models of set theory

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MathOverflow

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Models of set theory under embeddability

Consider the models of set theory under embeddability.

One model *embeds* into another, written $M \precsim N$, if there is $j : M \to N$ for which

$$x \in^M y \quad \longleftrightarrow \quad j(x) \in^N j(y).$$

In other words, $\langle M, \in^M \rangle$ is isomorphic to a substructure of $\langle N, \in^N \rangle$.

Incomparable models of set theory

It is extremely natural to inquire:

Question (Ewan Delanoy)

Exhibit two *incomparable* countable models of set theory, models that do not embed into each other.

$$M \not\hookrightarrow N \not\hookrightarrow M$$

The question was asked on math.SE, and several users posted suggested solutions.

Exhibiting incomparable models

There was an obvious strategy for producing incomparable models.

Let M be a tall thin model, and let M be a short, fat model.

The idea was: M is too tall to embed into N. And N is too fat to embed into M.

I tried hard to prove this, but could not make it work.

Eventually, I began to suspect that it just wasn't true...

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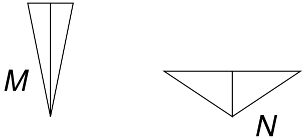

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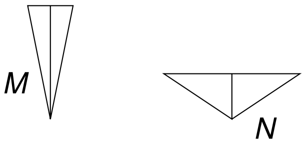

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Embeddability is linear

Main Theorem (Hamkins)

There are no incomparable countable models of set theory. Given any $\langle M, \in^M \rangle$ and $\langle N, \in^N \rangle$, one of them embeds into the other.

Thus, the countable models of set theory are linearly pre-ordered by embeddability.

Indeed, they are pre-well-ordered by embeddability in order type exactly $\omega_1 + 1$.

The proof proceeds from a graph-theoretic perspective, using graph universality and thinking of the models of set theory as acyclic directed graphs.

Only the height matters

The proof shows that embeddability of models of set theory reduces to the order-embeddability of their ordinals.

Theorem (Hamkins)

The following are equivalent for countable models of set theory.

1. $\langle M, \in^M \rangle$ embeds into $\langle N, \in^N \rangle$.
2. The ordinals of M embed into the ordinals of N.

So the short fat model embeds into the tall thin model!

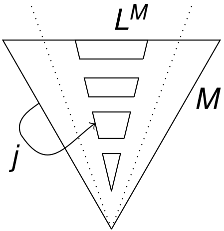
But also, any two countable models of set theory with the same ordinals are bi-embeddable.

Every model embeds into its own L

Theorem (Hamkins)

Every countable model of set theory $\langle M, \in^M\rangle$ is isomorphic to a submodel of its own constructible universe $\langle L^M, \in^M\rangle$.

In other words, there is an embedding $j : M \to L^M$, for which



$$x \in y \quad \longleftrightarrow \quad j(x) \in j(y).$$

Embedding into models of finite set theory

The embedding phenomenon arises even in finite set theory. Recall Ackermann's relation:

$$n\ E\ m \quad \text{iff} \quad n^{th} \text{ binary bit of } m \text{ is } 1.$$

It is an elementary exercise to see that $\langle \mathbb{N}, E\rangle \cong \langle \mathrm{HF}, \in\rangle$.

Theorem (Ressayre 1983)

For any nonstandard model $M \models \mathrm{PA}$ and any consistent c.e. set theory $T \supseteq \mathrm{ZF}$, there is $N \subseteq \langle \mathrm{HF}, \in\rangle^M$ with $N \models T$.

Thus, we find submodels of HF^M that satisfy ZFC. Incredible!

Ressayre uses partial saturation and resplendency to find a submodel of T.

A strengthening of Ressayre

Theorem (Hamkins)

If M is any nonstandard model of PA, then $\langle \mathrm{HF}, \in \rangle^M$ is universal for all countable acyclic binary relations.

In particular, every countable model of set theory is isomorphic to a submodel of HF^M.

Living inside HF^M, we believe every set is finite—it is the land of the finite—but by throwing some objects away, we arrive at a model of ZFC with large cardinals...

Universal structures

A structure M is *universal* for a class Δ of structures, if every structure in Δ embeds into M.

For example, the rational order $\langle \mathbb{Q}, < \rangle$ is universal for all countable linear orders.

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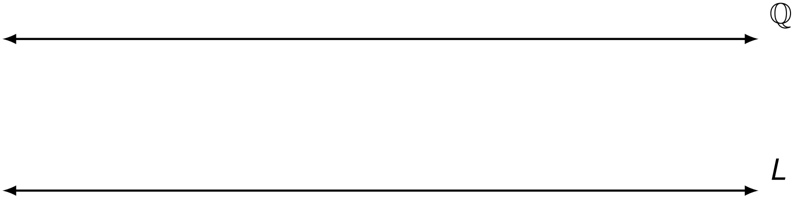

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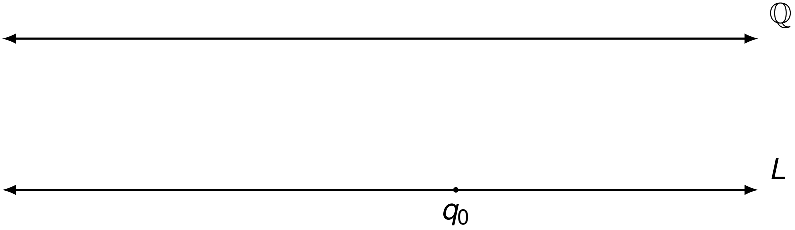

Enumerate the elements of your order L, and build the embedding in stages.

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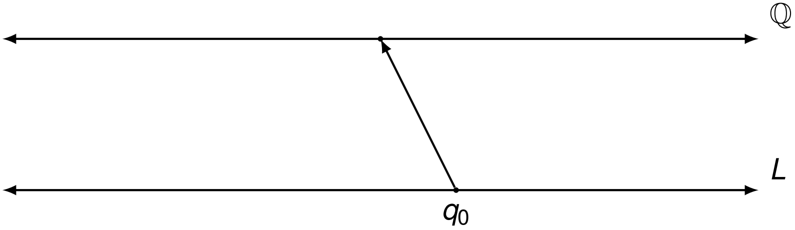

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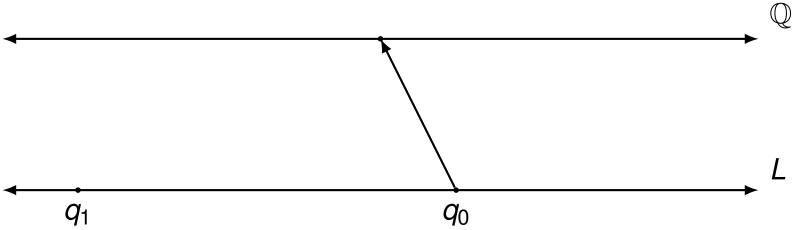

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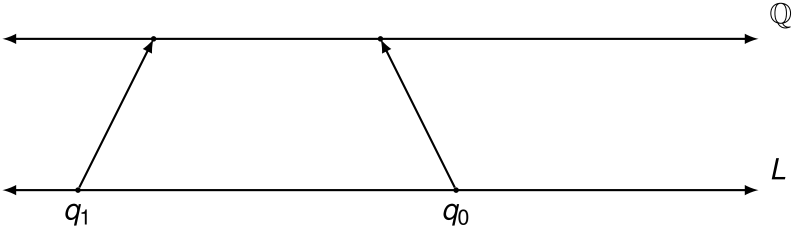

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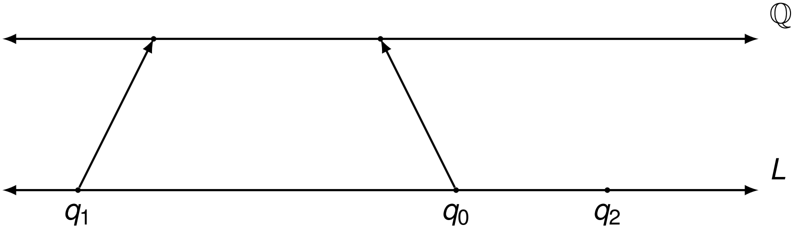

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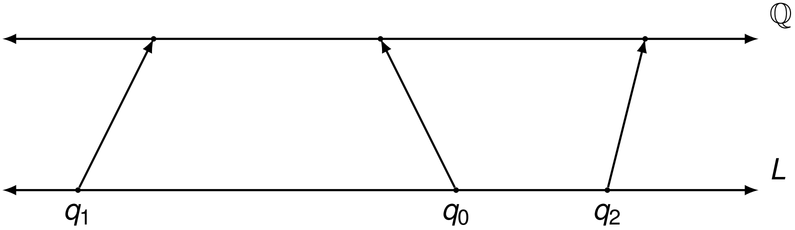

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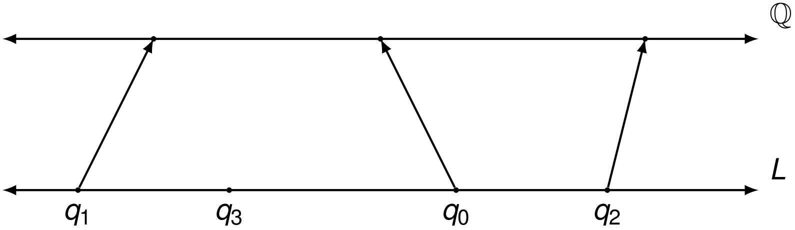

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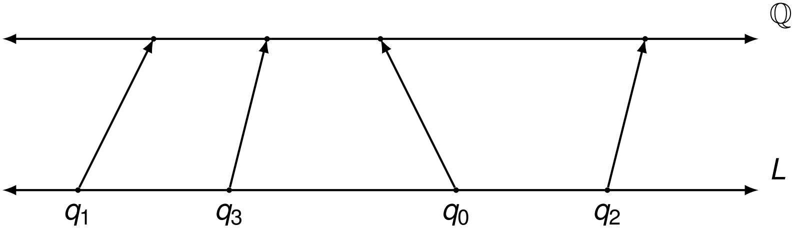

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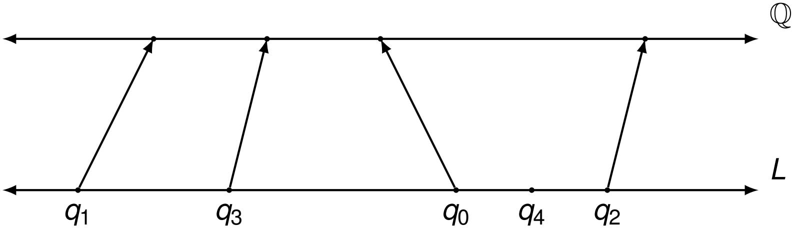

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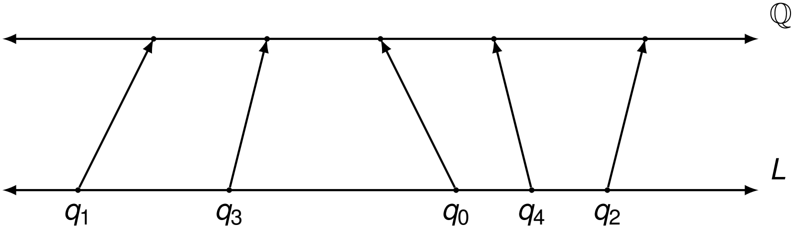

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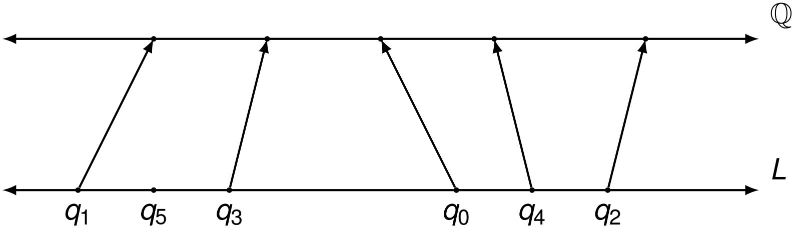

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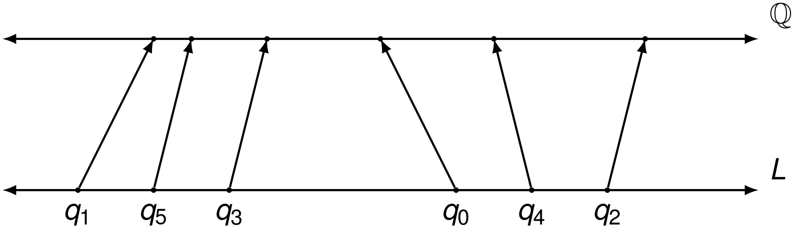

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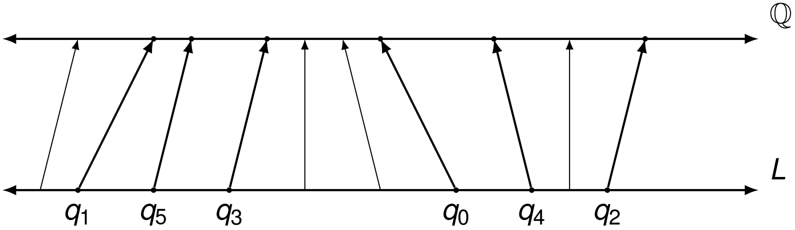

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Universal partial order

Can we construct a computable universal partial order?

Sure, it's easy...

Start with a single point.

Add new points relating to that point in all possible ways.

Keep doing that, adding finitely many points realizing types at each stage.

The resulting order is universal, by the "forth" part of Cantor's back-and-forth method.

This construction produces a *homogeneous* model, one for which finite partial automorphisms can always be extended.

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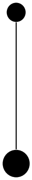

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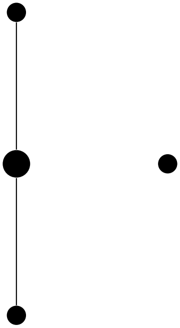

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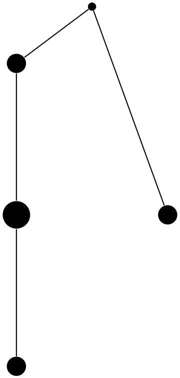

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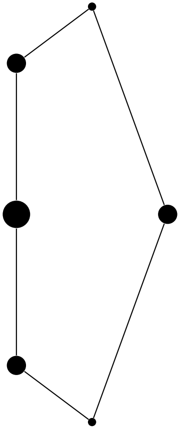

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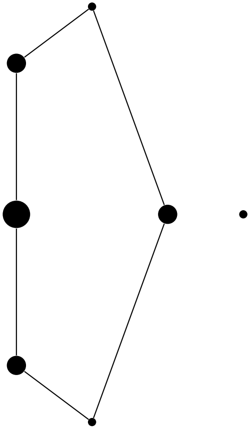

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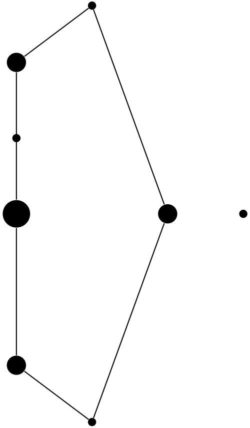

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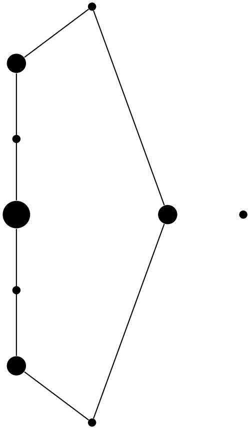

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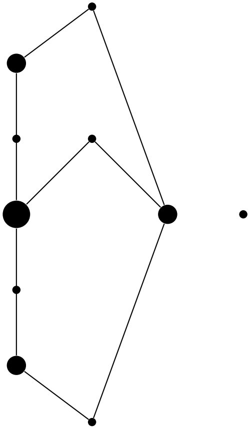

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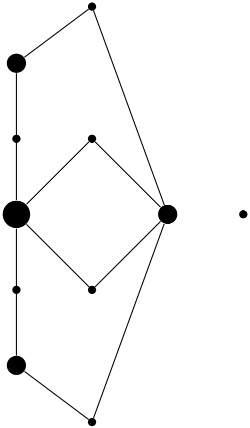

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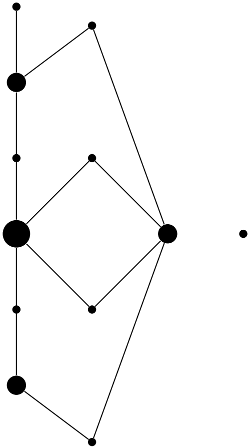

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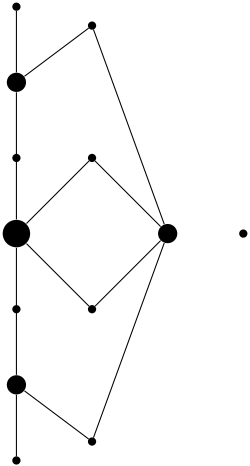

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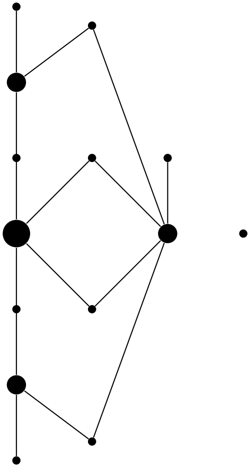

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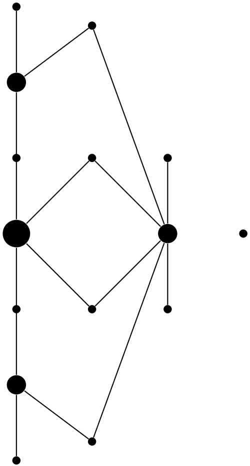

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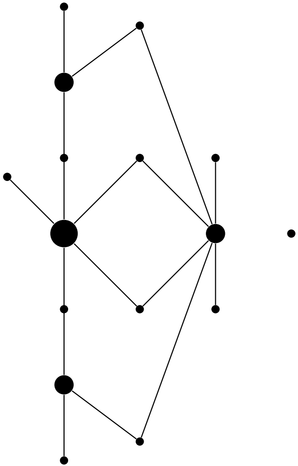

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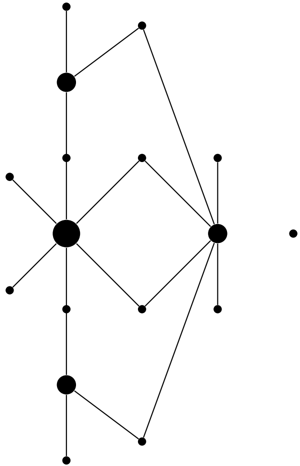

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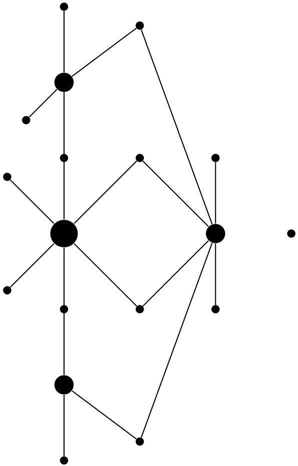

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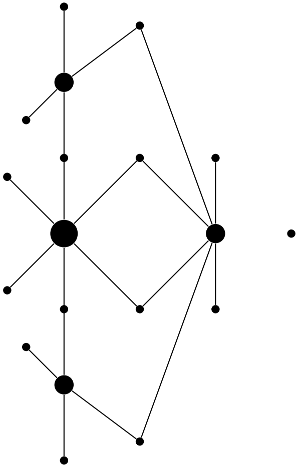

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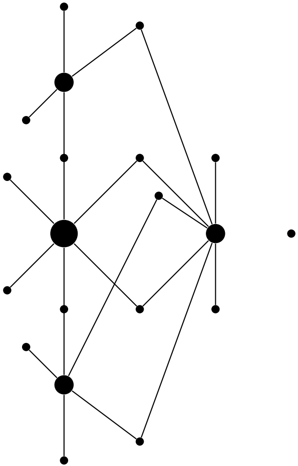

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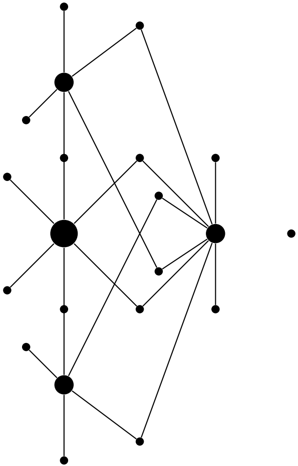

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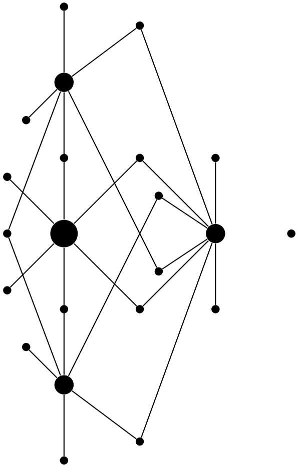

Countable random graph

For graphs, the construction produces the *countable random graph*.

The countable random graph is characterized by the *finite pattern property*: for any disjoint finite sets of nodes A, B, there is a node a connected to every node in A and to none in B.

A similar construction works with directed graphs, producing the countable random digraph, with a similar finite pattern property.

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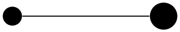

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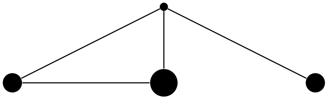

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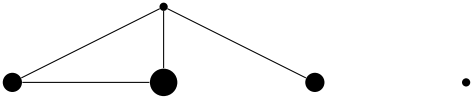

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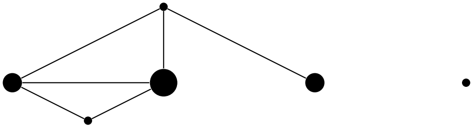

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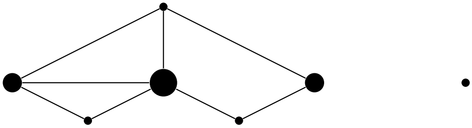

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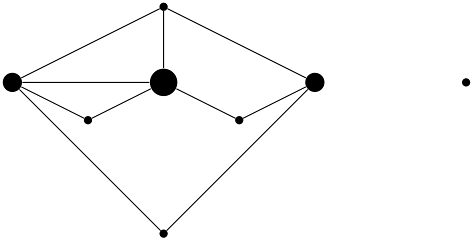

The countable random graph is characterized by the *finite pattern property*: for any disjoint finite sets of nodes A, B, there is a node a connected to every node in A and to none in B.

A similar construction works with directed graphs, producing the countable random digraph, with a similar finite pattern property.

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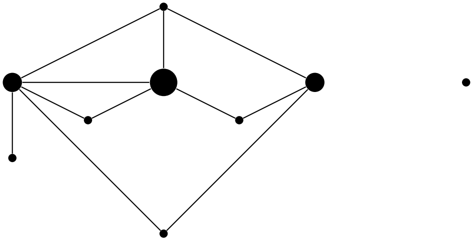

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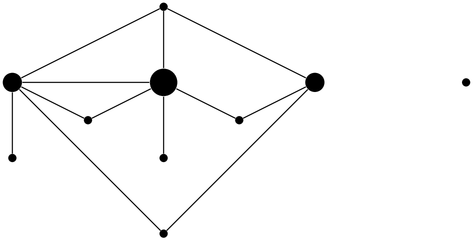

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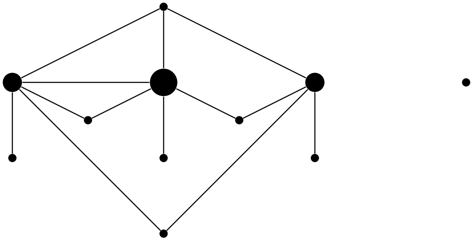

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Acyclic digraphs

A digraph is *acyclic* if it has no directed cycles.

(Every model of set theory $\langle M, \in^M \rangle$ is an acyclic digraph.)

May we undertake an analogous construction to produce a universal acyclic digraph?

No, the method doesn't work. We can't add new nodes in all possible ways, since this will create cycles.

The basic problem is a failure of *amalgamation*. New nodes, which are fine individually, cannot be amalgamated. The method is attempting to construct a homogeneous model, and there is no nontrivial homogeneous countable acyclic digraph.

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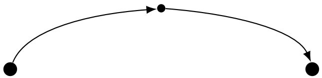

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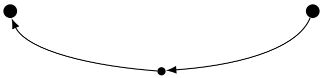

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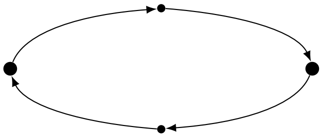

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Graded digraphs

The situation is better for *graded* digraphs.

A digraph $(G, \rightharpoonup)$ is $\mathbb{Q}$-*graded*, if there is $a \mapsto \alpha_a \in \mathbb{Q}$ such that

$$a \rightharpoonup b \quad \text{implies} \quad \alpha_a < \alpha_b.$$

More generally, a *graded digraph* is a digraph $(G, \rightharpoonup, \leq)$ accompanied by a linear pre-order $\leq$ on the nodes, such that $a \rightharpoonup b$ implies $a < b$.

- Every graded digraph is acyclic.
- Conversely, every countable acyclic digraph $(G, \rightharpoonup)$ can be $\mathbb{Q}$-graded: the transitive closure of $\rightharpoonup$ is a partial order, which extends to a linear order, which embeds into $\mathbb{Q}$.

The countable random $\mathbb{Q}$-graded digraph

Theorem

There is a countable $\mathbb{Q}$-graded homogeneous digraph Γ, universal for all countable $\mathbb{Q}$-graded digraphs. It is unique up to isomorphism and has a computable presentation.

Plentiful proof for the existence of this highly canonical object:

1. Γ is the Fraïssé limit of the finite $\mathbb{Q}$-graded digraphs.

2. Forcing construction: meet requirements (dense sets) to ensure the corresponding finite pattern property.

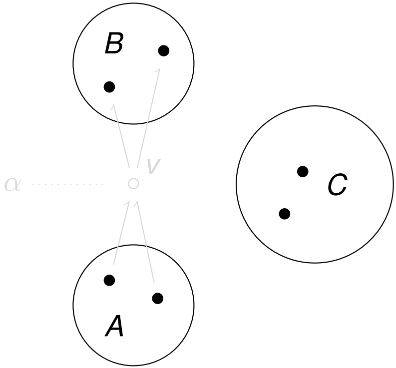
3. Computable presentation, adding nodes of each possible type at each stage, as before.

4. Probabilistic proof. Put infinitely many nodes with each value; connect edges with probability $\frac{1}{2}$. $\square$

Finite pattern property

The countable random $\mathbb{Q}$-graded digraph is characterized by:

For any disjoint finite sets of nodes *A*, *B*, *C*, any value α between *A* and *B*, there is a node *v* with:



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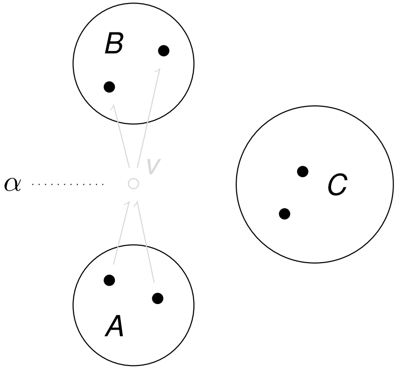

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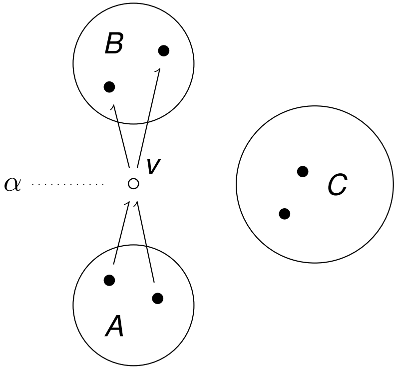


Realizing graphs as sets

Lemma

Every finite acyclic digraph $(G, \rightharpoonup)$ is isomorphic to a hereditarily finite set $(A, \in)$.

Proof.

Like Mostowski collapse, but graph may not be extensional. Let

$$\pi(y) = \{\, \pi(x) \mid x \rightharpoonup y \,\} \cup \{\{\emptyset, y\}\}\,,$$

and then show

$$x \rightharpoonup y \quad \Longleftrightarrow \quad \pi(x) \in \pi(y). \qquad \square$$

Actually, the proof works for well-founded acylic digraphs, realizing them as sets.

Proof of the embedding theorem

Theorem (Hamkins)

If M is any nonstandard model of PA, then every countable model of set theory arises as a submodel of $\langle \mathrm{HF}, \in \rangle^M$. Indeed, HF^M is universal for all countable acyclic binary relations.

Proof.

Let $M \models \mathrm{PA}$ be nonstandard. Build the countable random $\mathbb{Q}$-graded digraph Γ^M inside M. Let Γ_n be the n^{th} approximation for some nonstandard finite n. Since M thinks Γ_n is a finite acyclic digraph, it thinks $\langle \Gamma_n, \rightharpoonup \rangle \cong \langle A, \in^M \rangle$ for some $A \in \mathrm{HF}^M$. But since n is nonstandard, Γ_n includes the actual countable random $\mathbb{Q}$-graded digraph Γ, and so $\langle A, \in^M \rangle$ is universal for all countable acyclic binary relations. □

Surreal numbers

Construct the surreal numbers (Conway) by relentlessly, transfinitely filling cuts in what has been constructed so far.

The basic idea goes back to Hausdorff, who constructed saturated linear orders of arbitrarily large cardinality.

If A, B are sets of surreals already constructed, with $A < B$, then $\{ A \mid B \}$ is the surreal number filling the cut between A and B.

Define order $\{ X_L \mid X_R \} = x \leq y = \{ Y_L \mid Y_R \}$ if no obstacle prevents it, that is, $\nexists x_L \in X_L \, (y \leq x_L)$ and $\nexists y_R \in Y_R \, (y_R \leq x)$.

Define equivalence $x \simeq y \longleftrightarrow x \leq y \leq x$.

The class No of all surreals is homogeneous and universal for all class linear orders.

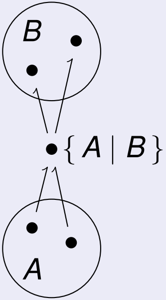
Hypnagogic digraph

Hypnagogic state: the dream-like sometimes hallucinatory state between sleeping and wakefulness.

Theorem

There is a surreals-graded class digraph $\langle \mathrm{Hg}, \rightharpoonup \rangle$ that is homogeneous and universal for all graded class digraphs.

Proof.



Proof. Canonical representation. Use surreal number *numerals* $\{ A \mid B \}$, but *do not quotient* by equivalence! Every node in A points at $\{ A \mid B \}$, and $\{ A \mid B \}$ points at every node in B. Grading value of node $\{ A \mid B \}$ is its surreal number value. $\square$

Connection with models of set theory

Main idea: every model of set theory $\langle M, \in^M\rangle$ is Ord^M-graded by von Neumann rank.

For any linear order ℓ, the restriction $\mathrm{Hg} \upharpoonright \ell$ is homogeneous and universal for all ℓ-graded digraphs.

Strategy: Given $\langle M, \in^M\rangle$, look at $(\mathrm{Hg} \upharpoonright \mathrm{Ord})^M$, which is universal for all Ord^M-graded digraphs.

Problem: $\langle \mathrm{Hg}, \rightharpoonup\rangle$ is not set-like, and so the modified Mostowski collapse lemma does not realize it in sets.

Worse: no model of ZFC has an Ord-graded set-like digraph with finite pattern property.

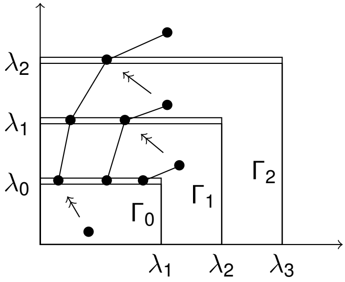
The main theorem

Main Theorem (Hamkins)

Every countable model of set theory $\langle M, \in^M\rangle$ is universal for all countable Ord^M-graded digraphs.

Proof ideas. Fix cofinal $\lambda_n \nearrow \mathrm{Ord}^M$. Let $\Gamma_n = (\mathrm{Hg} \upharpoonright \lambda_n + 1)^{V^M_{\lambda_{n+1}}}$.

Define *surrogate* digraph Θ, nodes are $\langle v_0, \ldots, v_n\rangle$, split parent and child roles.



Main theorem proof

The surrogate digraph Θ enjoys a *surrogate finite-pattern property*, which ensures universality.

Furthermore, $\Theta = \bigcup_n \Theta_n$ is the union of set-like Ord^M-graded digraphs Θ_n, which are each realized as sets in M. The surrogate relations ensure that nodes do not gain new children as n increases, and so $\langle \Theta, \twoheadrightarrow \rangle \cong \langle A, \in^M \rangle$ for some $A \subseteq M$.

Thus, $\langle M, \in^M \rangle$ is universal for all countable Ord^M-graded digraphs, as desired. $\square$

Conclusions

- One countable model M embeds in another N just in case $\mathrm{Ord}^M \lesssim \mathrm{Ord}^N$.
- Countable models of set theory with the same ordinals are bi-embeddable.
- Every countable model of set theory M embeds into its own constructible universe L^M.
- Countable nonstandard models of set theory are universal and mutually biembeddable.
- So there are $\omega_1 + 1$ many biembeddability classes for countable models of set theory.

Uncountable models

The embedding phenomenon fails for uncountable models.

Theorem (Fuchs, Gitman, Hamkins)

Assume that ZFC is consistent.

1. If $\diamondsuit$ holds, then there are 2^{ω_1} many pairwise incomparable ω_1-like models of ZFC.
2. There is an ω_1-like model $M \models \mathrm{ZFC}$ and an ω_1-like model $N \models \mathrm{PA}$ such that M does not embed into HF^N.
3. Relative to a Mahlo cardinal, it is consistent that there is a transitive ω_1-like model $M \models \mathrm{ZFC}$ that does not embed into its constructible universe L^M.

Internal embeddings

The main theorem showed that there are embeddings from any countable model of set theory M to its own constructible universe

$$j : M \to L^M$$

But can we find such embeddings j that are classes inside M? In particular,

Question (Hamkins)

Can there be a class embedding $j : V \to L$, if $V \neq L$?

This question is open, but we have some partial results.

Every countable set embeds into L

Theorem (Hamkins)

Every countable set A embeds into L: $\quad j : A \to L$

Proof.

Fix any countable A. Find ordinal θ with a grading function $r : A \to \theta$ with $a \in b \to r(a) < r(b)$. In L, build a θ-graded digraph $\langle \Gamma, \rightharpoonup, \rho \rangle$ with the finite-pattern property. Modified Mostowski collapse shows $\langle \Gamma, \rightharpoonup \rangle \cong \langle B, \in \rangle$ some $B \in L$. But it is also universal for all countable θ-graded digraphs. So $\exists j : A \to B$ and hence $j : A \to L$, as desired. $\square$

This is true even when $A \notin L$.

On embeddings of V into L

In joint work with myself, Yair Hayut, Menachem Magidor, W. Hugh Woodin, David Aspero:

Theorem

If there is $j : V \to L$, then the GCH *holds above* $\aleph_0$.

Theorem

If there is an embedding $j : V \to L$, then $0^\sharp$ does not exist.

Theorem

If there is an embedding $j : V \to L$, then the ground axiom holds; that is, the universe was not obtained by (set) forcing.

Forcing embeddings

Theorem

There is a notion of forcing, such that in the forcing extension $V[G]$, there are new reals, as well as an embedding

$$j : P(\omega)^{V[G]} \to P(\omega)^V.$$

We have a tentative argument for a similar phenomenon at higher cardinals, assuming large cardinals.

The main question remains open: Is it possible that $j : V \to L$ when $V \neq L$?

Embeddings of V

The model-theoretic embedding concept is weaker than the usual set-theoretic embedding concept. Compare with the Kunen inconsistency:

Theorem

There is a nontrivial definable embedding $j : V \to V$.

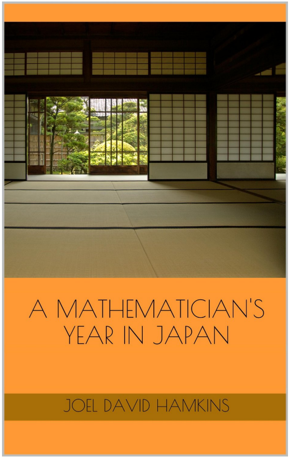
Proof.

Let

$$j(y) = \{ j(x) \mid x \in y \} \cup \{\{\emptyset, y\}\} .$$

It is not difficult to prove $x \in y \longleftrightarrow j(x) \in j(y)$. □

The Kunen inconsistency rules out nontrivial cofinal Δ_0-elementary embeddings.



A Mathematician's year in Japan.
Joel David Hamkins
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"Glimpse into the life of a professor of logic as he fumbles his way through Japan. A Mathematician's Year in Japan is a lighthearted, though at times emotional account of how one mathematician finds himself in a place where everything seems unfamiliar, except his beloved research on the nature of infinity, yet even with that he experiences a crisis."

Thank you. Slides available at http://jdh.hamkins.org.