Reducibilities as refinements of the randomness hierarchy

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Question

Let $A, B \in 2^\omega$.

Suppose B is "more random than" A,

and A is "random".

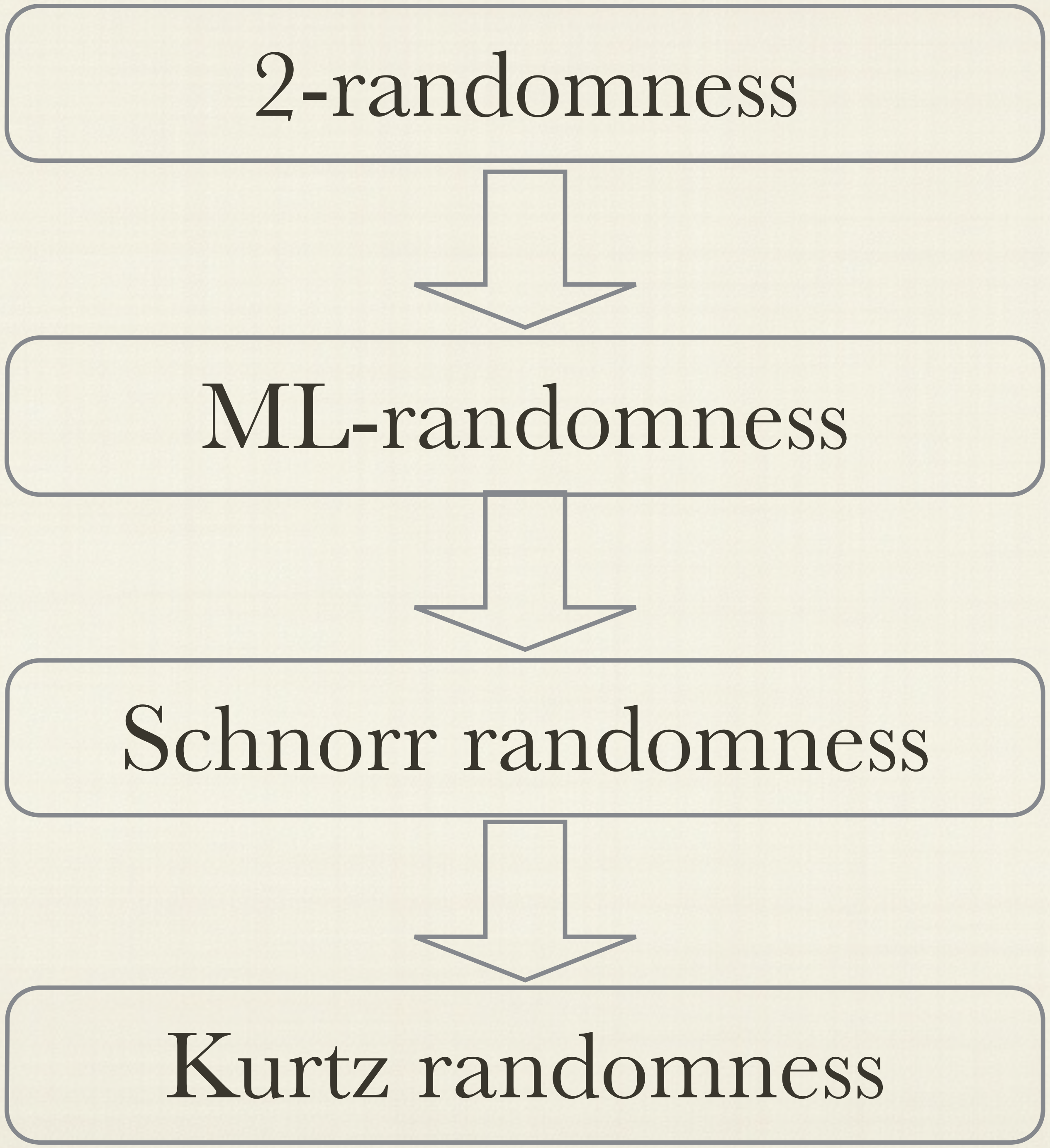
Then, should B be "random"?

Overview

- What is “random”? - Randomness hierarchy
  What is “more random”? - Reducibility
- Consistent of for some pairs and not for others.
- 3-randomness via complexity

Two measures of randomness

Randomness hierarchy



ML-randomness

Theorem (Levin 1973, Schnorr 1973)

A set $X \in 2^\omega$ is ML-random if and only if

$$\exists c \forall n \; K(X \restriction n) > n - c.$$

K-reducibility

Definition

Let $A, B \in 2^\omega$. A is K-reducible to B, denoted by $A \leq_K B$, if

$$K(A \upharpoonright n) \leq K(B \upharpoonright n) + O(1).$$

Its intuitive meaning is that, A is more compressible than B, that is, B is more random than A.

Observation

If $A \leq_K B$ and A is ML-random, then B is ML-random.

Two measures of randomness

- Randomness hierarchy
- K-reducibility

- Are there consistent with each other?
- Can we think K-reducibility as a refinement of the randomness hierarchy?

Inconsistency

Theorem (Theorem 7.4.11 in Nies' book)

There exists a computably random set X such that, for every computable order h, we have $K(X \upharpoonright n \mid n) \leq h(n) + O(1)$.

Observation

Consider X above and a ML-random real Y. Then, $X \leq_K Y \oplus \emptyset$ and X is Schnorr random, but $Y \oplus \emptyset$ is not Schnorr random.

Consistency

Definition

We say that a reducibility $\leq_r$ is consistent with a randomness notion $\mathcal{R}$ if the following statement holds: For all sets A and B, if $A \leq_r B$ and $A \in \mathcal{R}$, then $B \in \mathcal{R}$.

Observation

$\leq_K$ is consistent with ML-randomness, but not consistent with Schnorr randomness or Kurtz randomness.

	WR	Schnorr	MLR	2R	n-R
plain (C)	No (Nies)	No (Nies)	Yes (Gács, Miller-Yu)	Yes (Miller, Nies-Stephan-Terwijn)	Yes (Miller-Yu)
prefix-free (K)	No (Nies)	No (Nies)	Yes (Levin, Schnorr)	Yes (Miller)	Yes (Miller-Yu)
total	Yes (Bienvenu-Merkle)	Yes (M.)	Yes (M.)	Yes (Nies)	Yes (M.)
Prefix-free decidable	Yes (Bienvenu-Merkle)	Yes (Bienvenu-Merkle)	Yes (Bienvenu-Merkle)	Yes (M.)	?
c.m.m. (Schnorr)	Yes (Downey-Griffiths-Reid)	Yes (Downey-Griffiths)	?	Yes (M.)	? (in progress)

Schnorr reducibility is consistent with 2-randomness

Schnorr randomness

Definition (Downey and Griffiths)

A computable measure machine is a prefix-free machine M such that $\sum_{\sigma \in \mathrm{dom}(M)} 2^{-|\sigma|}$ is a computable real.

Theorem (Downey and Griffiths)

A set $A \in 2^\omega$ is Schnorr random if and only if $K_M(A \upharpoonright n) > n - O(1)$ for every c.m.m. M.

Schnorr reducibility

Definition (Downey and Griffiths)

$A \leq_{Sch} B$ if, for every c.m.m. M, there exists a c.m.m. N such that

$$K_N(A \upharpoonright n) < K_M(B \upharpoonright n) + O(1).$$

Theorem (Miller)

A is 2-randomness if and only if

$$K(A \upharpoonright n) > n + K(n) - O(1)$$

for infinitely many n.

Question

A is 2-randomness if and only if, for every c.m.m. M,

$$K_M(A \upharpoonright n) > n + ??(n) - O(1)$$

for infinitely many n.

$$Q_M(\sigma) = \mu(\llbracket\{\tau \ : \ M(\tau)\downarrow = \sigma\}\rrbracket)$$

Theorem (Coding theorem)

$$K(\sigma) = -\log Q(\sigma).$$

Definition

$$R^1_M(n) = -\log \mu(\llbracket\{\tau \ : \ |M(\tau)\downarrow| = n\}\rrbracket)$$

Theorem (M.)

$$K(n) = R^1_U(n) \pm O(1).$$

Extended counting theorem

Theorem (Counting theorem)

$$|\{\sigma \ : \ |\sigma| = n \wedge K(\sigma) \leq n + K(n) - r\}| \leq 2^{n-r+O(1)}$$

Theorem (Extended counting theorem, M.)

$$|\{\sigma \ : \ |\sigma| = n \wedge K_M(\sigma) \leq n + R^1_M(n) - r\}| \leq 2^{n-r+O(1)}$$

2-randomness via c.m.m.

Theorem (M.)

X is 2-random if and only if, for every c.m.m. M, we have

$$K_M(X \upharpoonright n) \geq n + R^1_M(n) - O(1)$$

for infinitely many n.

Corollary

Schnorr reducibility is consistent with 2-randomness (and thus so is dm-reducibility).

3-randomness via complexity

Motivation

- Schnorr reducibility and n-randomness?
- 2-randomness version of vL-reducibility!!

Theorem (Miller and Yu)

$X \oplus Z$ is ML-random if and only if

$$K(X \upharpoonright (Z \upharpoonright n)) > (Z \upharpoonright n) + n - O(1).$$

Theorem (M.)

X is 2-Z-random if and only if $C(X \upharpoonright (Z \upharpoonright n)) > Z \upharpoonright n - O(1)$ for infinitely many n.

Corollary (M.)

X is 3-random if and only if $C(X \upharpoonright (\Omega \upharpoonright n)) > \Omega \upharpoonright n - O(1)$ for infinitely many n.

Theorem (M.)

$X \oplus Z$ is 2-random if and only if $K(X \upharpoonright (Z \upharpoonright n)) > (Z \upharpoonright n) + n + K(n) - O(1)$ for infinitely many n.

Summary

- Weak reducibilities are consistent with strong randomness notions.
- Complexity of a set can be measured by the set of lengths where the complexities are maximal.
- Is Schnorr reducibility consistent with n-randomness?
- Is Schnorr reducibility consistent with ML-randomness?