Intuitionistic Provability, Classical Validity and Situation-Dependent Propositions — A Consideration based on Godel's Modal Embedding

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It is well-known that there exists a sound and faithful embedding of the sequents deducible in intuitionistic logic (IL) into those deducible in the modal logic S4: for any sequent S of IL, IL proves S if and only if S4 proves the modal translation (i.e. the so-called Godel translation) of S. Moreover, we can show that there exists a more exact model theoretic correspondence between IL propositions (i.e. IL formulae) and their S4 translations: for any Kripke model M of S4, we can construct i) a mapping f on the set W of possible worlds in M and ii) a Kripke model L of IL that is defined in terms of the mapping f, such that for any S4 proposition P that is the modal translation of some IL proposition Q (i.e. P is the Godel translation of Q), (1) f not only strictly preserves the truth (i.e. P is true in M iff. Q is true in L), (2) but also maps the truth set of P in M (i.e. the set of the words in W in which P is true) exactly into that of Q in L (i.e. the set of possible states of L in which Q is true). The converse mapping from IL Kripke models to S4 Kripke models can be also (rather trivially) constructed.

One might say these facts justify the claim that intuitionistic propositions be literally construed in accordance with their modal translations: for example, for any IL primitive proposition q, we could construed it as stating that the corresponding S4 primitive proposition (usually this latter proposition is itself symbolized by "q", but we are free to assign to it any S4 primitive proposition whatever) is "necessary", since the Godel translation of q is the necessitation of the corresponding S4 primitive proposition. Note that the notion of necessity in question here is nothing other than that of the intuitionistic provability.

Now, similar observations could be made with respect to classical logic (Cl) and the modal logic S5: (1) the modal translation embeds sequents deducible in CL into those deducible in S5 in a sound and faithful way and (2) there exists an exact model theoretic correspondence between CL propositions and their S5 translations. As might easily be seen, in this case the relevant notion of necessity is nothing other than that of classical validity: for example, the CL primitive proposition q could be construed as stating that the corresponding S5 primitive proposition is classically valid.

These considerations induce the following question: what are the (primitive) propositions of S4 and S5? We attempt to show that they are in general to be identified with situation-dependent propositions (some authors say they are the propositions seen from the “internal” or “local” point of view) and that this construal also throw some light on the relationships between intuitionistic and classical formal arithmetical theories.

Main Question: What is Intuitionistic Logic/Classical Logic a logic of?

[0] GMT translation:

$g(p) = \Box p$

$g(A \wedge B) = g(A) \wedge g(B)$

$g(A \vee B) = g(A) \vee g(B)$

$g(A \to B) = \Box(g(A) \to g(B))$

$g(\neg A) = \Box \neg g(A)$

e.g. $g(p \wedge q \to (r \to s \vee t)) = \Box(\Box p \wedge \Box q \to \Box(\Box r \to \Box s \vee \Box q)$

[1] Classical Logic LK

(1) Axiom $A \vdash A$

(2) Structural Rules

・EL $\dfrac{\Gamma 1,\ A,\ B,\ \Gamma 2 \vdash \Delta}{\Gamma 1,\ B,\ A,\ \Gamma 2 \vdash \Delta}$ ・ER $\dfrac{\Gamma \vdash \Delta 1,\ A,\ B,\ \Delta 2}{\Gamma \vdash \Delta 1,\ B,\ A,\ \Delta 2}$

・Cut $\dfrac{\Gamma \vdash \Delta,\ A \qquad \Theta,\ A \vdash \Xi}{\Gamma,\ \Theta \vdash \Delta,\ \Xi}$

・WL $\dfrac{\Gamma \vdash \Delta}{\Gamma,\ A \vdash \Delta}$ ・WR $\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta,\ A}$

・CL $\dfrac{\Gamma 1,\ A,\ A,\ \Gamma 2 \vdash \Delta}{\Gamma 1,\ A,\ \Gamma 2 \vdash \Delta}$ ・CR $\dfrac{\Gamma \vdash \Delta,\ A,\ A}{\Gamma \vdash \Delta 1,\ A}$

(3) Inference Rules

・∧L $\dfrac{\Gamma,\ A \vdash \Delta}{\Gamma,\ A \wedge B \vdash \Delta}$ ・∧R $\dfrac{\Gamma \vdash \Delta,\ A \qquad \Gamma \vdash \Delta,\ B}{\Gamma \vdash \Delta,\ A \wedge B}$

・∨L $\dfrac{\Gamma,\ A \vdash \Delta \qquad \Gamma,\ B \vdash \Delta}{\Gamma,\ A \vee B \vdash \Delta}$ ・∨R $\dfrac{\Gamma \vdash \Delta,\ A}{\Gamma \vdash \Delta,\ A \vee B}$

・→L $\dfrac{\Gamma \vdash \Delta,\ A \qquad \Theta,\ B \vdash \Xi}{\Gamma,\ \Theta,\ A \to B \vdash \Delta,\ \Xi}$ ・→R $\dfrac{\Gamma,\ A \vdash \Delta,\ B}{\Gamma \vdash \Delta,\ A \to B}$

・¬L $$\frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta}$$ ・¬R $$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

・TR $$\vdash T$$

[2] Intuitionistic Logic LJ

(1) Axiom $A \vdash A$

(2) Structural Rules

・EL $$\frac{\Gamma 1, A, B, \Gamma 2 \vdash C}{\Gamma 1, B, A, \Gamma 2 \vdash C}$$

・Cut $$\frac{\Gamma \vdash A \qquad \Theta, A \vdash B}{\Gamma, \Theta \vdash B}$$

・WL $$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}$$

・CL $$\frac{\Gamma 1, A, A, \Gamma 2 \vdash \Delta}{\Gamma 1, A, \Gamma 2 \vdash \Delta}$$

(3) Inference Rules

・∧L $$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}$$ ・∧R $$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash \Delta, A \wedge B}$$

・∨L $$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$ ・∨R $$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$

・→L $$\frac{\Gamma \vdash \Delta, A \qquad \Theta, B \vdash \Xi}{\Gamma, \Theta, A \rightarrow B \vdash \Delta, \Xi}$$ ・→R $$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$$

・¬L $$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash \Delta}$$ ・¬R $$\frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}$$

・⊥L $$\bot \vdash$$

[3] Modal Sequent Calculi

K : $$\frac{\Gamma \Rightarrow A, \Delta}{\Box\Gamma \Rightarrow \Box A, \Diamond\Delta}$$ (Nec : $$\frac{\Rightarrow A}{\Rightarrow \Box A}$$)

K^d : $$\frac{\Gamma, A \Rightarrow \Delta}{\Box\Gamma, \Diamond A \Rightarrow \Diamond\Delta}$$

T□ : $$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}$$

T◇ : $$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Diamond A, \Delta}$$

4□ : $$\frac{\Box\Gamma \Rightarrow A, \Diamond\Delta}{\Box\Gamma \Rightarrow \Box A, \Diamond\Delta}$$

4◇ : $$\frac{\Box\Gamma, A \Rightarrow \Diamond\Delta}{\Box\Gamma, \Diamond A \Rightarrow \Diamond\Delta}$$

5□ : $$\frac{\Box\Gamma \Rightarrow A, \Box\Delta}{\Box\Gamma \Rightarrow \Box A, \Box\Delta}$$

5◇ : $$\frac{\Diamond\Gamma, A \Rightarrow \Diamond\Delta}{\Diamond\Gamma, \Diamond A \Rightarrow \Diamond\Delta}$$