# On the reals which can be random

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# What is a random real

A random real must pass all the *performable* statistical tests.

# Martin-Löf-randomness

**Definition**

- A Martin-Löf test is an r.e. sequence of open sets $\{U_n\}_{n\in\omega}$ so that $\forall n \mu(U_n) < 2^{-n}$.
- A real $x$ *passes* a Martin-Löf test $\{U_n\}_{n\in\omega}$ if $x \notin \bigcap_n U_n$.
- A real $x$ is *Martin-Löf-random* (or 1-random) if it passes all Martin-Löf tests.

# Distributions of random reals

A Martin-Löf random real $x$ can

- $<_T \emptyset'$;
- range over $\geq_T \emptyset'$;
- Turing incomparable with $\emptyset'$.

# Probability measure

**Definition**

A measure $\rho$ over $2^\omega$ is *probability measure* if

1. $\rho(\emptyset) = 1$; and
2. For any $\sigma \in 2^{<\omega}$, $\rho(\sigma) = \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1)$.

# Continuous measure

**Definition**

Given a measure $\rho$, a real $x$ is an *atomic* respect to $\rho$ if $\rho(\{x\}) > 0$.

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A measure $\rho$ is *continuous* if $\forall x \rho(\{x\}) = 0$.

# Representing measures

A measure $\rho$ can be represented by $\{(p, q, \sigma) \in \mathbb{Q}^2 \times 2^{<\omega} \mid p \leq \rho(\sigma) \leq q\}$.

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A measure $\rho$ is recursive if its representation is recursive.

# Randomness under general probability measures

For a fixed probability measure $\rho$, we may define Martin-Löf randomness respect to $\rho$.

# *NCR*

**Definition**

A real $x$ is never continuous random (or *NCR*), if $x$ cannot be random respect to any continuous measure.

# *NCR* is countable

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**Theorem (Reimann and Slaman)**

*Every real in NCR is hyperarithmetic.*

The proof is based on Woodin's result that every nonhyperarithmetic real *tt*-cups a real $z$ to $\mathcal{O}^z$, the hyperjump relative to $z$.

# Higher randomness

**Definition**

1. A real $x$ is $\Delta^1_1$-random if it does not belong to any $\Delta^1_1$-null set.
2. A real $x$ is $\Pi^1_1$-random if it does not belong to any $\Pi^1_1$-null set.

# Some basic facts

**Theorem (Sacks)**

*$\{x \mid x \geq_h \mathcal{O}\}$ is a $\Pi^1_1$ null set.*

**Theorem (Kechris; Stern; Hjorth and Nies)**

*There is a largest $\Pi^1_1$-null set.*

**Theorem (Stern; Chong, Nies and Yu)**

*A real $x$ is $\Pi^1_1$-random iff $x$ is $\Delta^1_1$-random and $\omega_1^x = \omega_1^{\mathrm{CK}}$ iff $x$ is $\Delta^1_1$-random and every function $\Delta^1_1$ in $x$ is dominated by a $\Delta^1_1$-function.*

# On $NCR_{\Pi^1_1}$

**Theorem (Chong and Yu)**

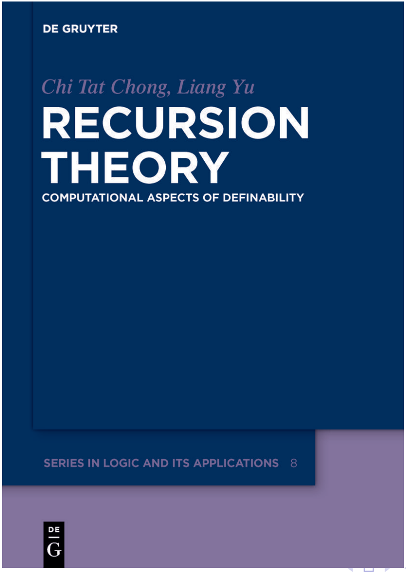
$NCR_{\Pi^1_1} = \{x \mid x \in L_{\omega_1^x}\}$.

**Proof.**

- $NCR_{\Pi^1_1}$ is a $\Pi^1_1$-set.
- $NCR_{\Pi^1_1}$ does not contain a perfect subset.
- If $x \in L_{\omega_1^x}$, then for any continuous measure $\rho$, either $\rho \geq_h x$ or $x \oplus \rho \geq_h \mathcal{O}^\rho$.

$\square$

# To fully understand these facts



# $L$-randomness

**Definition**

A real $x$ is $L$-random if for any $L$-coded sequence open sets $\{U_n\}_{n\in\omega}$ with $\forall n\mu(U_n) < 2^{-n}$, $x \notin \bigcap_n U_n$.

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So an $L$-random real is exactly a Solovay real.

# $NCR_L$

**Proposition (Yu and Zhu)**

- *$NCR_L$ is a $\Pi^1_3$-set.*
- *If $NCR_L \neq 2^\omega$, then it is not $\Pi^1_2$.*
- *If $x$ is $L$-random and $y \in L[x] \setminus L$, then $y \notin NCR_L$.*
- *If $V = L[r]$ for some $L$-random real $r$, then $NCR_L$ is a proper $\Sigma^1_2$-set.*
- *If for any real $x$, $(2^\omega)^{L[x]}$ is countable, then*
  - *$NCR_L$ does not contain a perfect subset.*
  - *$NCR_L$ is $\Sigma^1_2$ if and only if $NCR_L \subseteq L$.*

The third item follows from a set theoretic version of Demuth Theorem.

# Under *PD*

**Proposition**

*Every $\Pi^1_2$-singleton belongs to $NCR_L$. Actually if $A$ is a countable $\Pi^1_2$-set, then $A \subseteq NCR_L$.*

**Proof.**

By Shoenfield absoluteness. $\square$

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*$0^\sharp$ is a $\Pi^1_2$-singleton.*

**Theorem (Friedman)**

*There is a nonconstructible $\Pi^1_2$-singleton $x <_L 0^\sharp$.*

By Friedman, there exists a non-$\Pi^1_2$-singleton belonging to a countable $\Pi^1_2$-set.

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$P_2 = \{x \mid \forall y(\kappa^x \leq \kappa^y \implies x \leq_L y)\}$.

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Note that $x \in P_2 \implies x^\sharp \in P_2$.
$0, 0^\sharp, (0^\sharp)^\sharp, \cdots \in P_2$.

# $P_2 \subseteq NCR_L$.

**Proposition (Yu and Zhu)**

*$P_2 \subseteq NCR_L$.*

**Proof.**

If $x$ is $L$-random respect to $\rho$, then $\kappa^x \leq \kappa^\rho$. So $x \in L[\rho]$, a contradiction. $\square$

# $Q_3$

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By the previous result, $Q_3 \not\subseteq NCR_L$.

# $NCR_L \subseteq Q_3$ (I)

**Theorem (Yu and Zhu)**

*$NCR_L \subseteq Q_3$.*

**Lemma**

*For any real $x$, there is a real $y \geq_T x$ so that there is a continuous measure $\rho \leq_T y$ so that $y$ is $L$-random respect to $\rho$.*

Let

$$B = \{y \mid \exists \rho \leq_T y(\rho \text{ is continuous and } y \text{ is } L\text{-random respect to } \rho)\}.$$

Then $B$ is a $\Pi^1_2$ set and has cofinally many $L$-degrees.

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$B$ is co-uncountable.

# $NCR_L \subseteq Q_3$ (II)

Let $D = \{y_0 \mid \forall y(y \geq_T y_0 \rightarrow y \in B)\}$ be a nonempty $\Pi^1_2$-set.
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The relativized version is read as that for any real $z$, the set $B_z = \{y \mid \exists \rho \leq_T y \oplus z(\rho$ is continuous and $y$ is $L[z]$-random respect to $\rho)\}$ contains an upper cone with the base $y_{z,3}$.

# $NCR_L \subseteq Q_3$ (III)

A higher version of Posner-Robinson Theorem.

**Theorem (Woodin)**

*If $x \notin Q_3$, then there is a real $z$ so that $x \oplus z \geq_{tt} y_{z,3}$.*

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Applying Demuth's technique, we have that $x$ is $L[z]$-random respect to some continuous measure $\rho_0 \leq_L z \oplus \rho$.

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Applying Demuth's technique, we have that $x$ is $L[z]$-random respect to some continuous measure $\rho_0 \leq_L z \oplus \rho$.

So $NCR_L \subseteq Q_3$.

# Some questions

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1. $NCR_L = \bigcup\{A \mid A$ *is a countable and* $\Pi^1_2\}$*?*
2. *Is* $NCR_L$ *cofinal (in the L-degree sense) in* $Q_3$*?*

Thanks