Axiom schema of Markov's principle preserves disjunction and existence properties

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Introduction: Disjunction and Existence Properties

"Hallmarks" of constructivity of intuitionistic logic $\mathbf{H}_*$:

Fact

$\mathbf{H}_*$ has the disjunction property (DP);
for every $A \vee B$: $\mathbf{H}_* \vdash A \vee B \Rightarrow \mathbf{H}_* \vdash A$ or $\mathbf{H}_* \vdash B$.

$\mathbf{H}_*$ has the existence property (EP);
for every $\exists x A(x)$: $\mathbf{H}_* \vdash \exists x A(x) \Rightarrow$ there exists a v such that $\mathbf{H}_* \vdash A(v)$.

N.B. $A(v)$ should be taken as a formula congruent to A free from collision of variables.

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$\mathbf{H}_* + A$: the logic obtained from $\mathbf{H}_*$ by adding the axiom schema A.
There are schmemata A such that $\mathbf{H}_* + A$ enjoys both of DP and EP.

We are interested in such schemata (i.e., $\mathbf{H}_* + A$ still enjoys DP and EP) in the setting of Intermediate Predicate Logics,
particularly in those schemata related to constructive theories.

N.B. $A(v)$ should be taken as a formula congruent to A free from collision of variables.

Markov's Principle and Limited Principle of Omniscience

In the setting of intermediate Predicate Logics, we consider:

Axiom schema of Markov's principle:

$$MP: \quad \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists x A(x) \rightarrow \exists x A(x).$$

Axiom schema of the limited principle of omniscience:

$$LPO: \quad \forall x(A(x) \vee \neg A(x)) \rightarrow \exists x A(x) \vee \neg\exists x A(x),$$

Both principles enlarge the concept of constructivity, particularly the concept of $\exists$ from that of intuitionistic logic $\mathbf{H}_*$.
However, still we have:

Theorem

$\mathbf{H}_* + MP$ and $\mathbf{H}_* + LPO$ enjoy DP and EP.
That is, MP and LPO preserve DP and EP.

Harrop-DP and Harrop-EP

Definition

A formula is said to be a Harrop-formula (H-formula) if every strictly positive subformula is neither of the form $A \vee B$ nor $\exists x A(x)$.

Theorem (Harrop)

$\mathbf{H}_*$ *has the H(arrop)-DP and the H(arrop)-EP, i.e., for any H-formula H,*
$\mathbf{H}_* \vdash H \to A \vee B \Rightarrow \mathbf{H}_* \vdash H \to A$ *or* $\mathbf{H}_* \vdash H \to B$,
$\mathbf{H}_* \vdash H \to \exists x A(x) \Rightarrow \mathbf{H}_* \vdash H \to A(v)$ *for some* v.

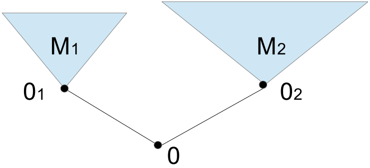
Theorem

$\mathbf{H}_* + MP$ *and* $\mathbf{H}_* + LPO$ *enjoy H-DP and H-EP.*
That is, MP and LPO preserve H-DP and H-EP.

Pointed Join of Kripke Models

Definition

- $\mathbf{M}_1$, $\mathbf{M}_2$: Kripke frames with the least elements 0_1 and 0_2, resp., such that the domains at 0_1 and 0_2 coincide with $V(= D_1(0_1) = D_2(0_2))$.
  A Kripke frame $\mathbf{M}$ is said to be the pointed join frame of $\mathbf{M}_1$ and $\mathbf{M}_2$, if $\mathbf{M} = \{(0, V)\} \uparrow \mathbf{M}_1 \oplus \mathbf{M}_2$ with a fresh least element 0.
- $(\mathbf{M}_1, \models_1)$, $(\mathbf{M}_2, \models_2)$: Kripke models with $V = D_1(0_1) = D_2(0_2)$.
  A Kripke model $(\mathbf{M}, \models)$ is said to be a pointed join model of $(\mathbf{M}_1, \models_1)$ and $(\mathbf{M}_2, \models_2)$,
  if $\mathbf{M}$ is the pointed join frame of $\mathbf{M}_1$ and $\mathbf{M}_2$, and the restrictions of $\models$ to $\mathbf{M}_1$ and $\mathbf{M}_2$ are $\models_1$ and $\models_2$, resp.



Axiomatic Truth and its Preservation

Definition

A formula A is said to be axiomatically true in a Kripke model $(\mathbf{M}, \models)$, if universal closures of all of substitution instances of A are true in $(\mathbf{M}, \models)$.

Lemma

If A preserves its axiomatic truth in the construction of pointed join models, i.e., satisfies the following:

- If A is axiomatically true in Kripke models $(\mathbf{M}_1, \models_1)$ and $(\mathbf{M}_2, \models_2)$ with $V = D_1(0_1) = D_2(0_2)$, then A is still axiomatically true in any pointed join model of $(\mathbf{M}_1, \models_1)$ and $(\mathbf{M}_2, \models_2)$,

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then $\mathbf{H}_* + A$ preserves H-DP and H-EP.

Theorem

MP and LPO have this property.
Hence, MP and LPO preserve H-DP and H-EP.

Another Phenomenon: Prawitz-Doorman EP

Definition

A formula is said to be a weak Harrop-formula (wH-formula) if every strictly positive subformula is not of the form $\exists x A(x)$.

Theorem (Prawitz, Doorman)

$\mathbf{H}_*$ has the Prawitz-Doorman EP , i.e.,
for any wH-formula H,
$\mathbf{H}_* \vdash H \to \exists x A(x) \Rightarrow$
there exist finitely many $v_1, \ldots, v_n$ in the vocabulary of $H \to \exists x A(x)$
such that $\mathbf{H}_* \vdash H \to A(v_1) \vee \cdots \vee A(v_n)$.

Prawitz proved EP of $\mathbf{H}_*$ by showing DP and the Prawitz-Doorman EP.

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Proposition

$\mathbf{H}_* + MP$ and $\mathbf{H}_* + LPO$ fail to have the Prawitz-Doorman EP.
That is, MP and LPO do not preserve the Prawitz-Doorman EP.

Concluding Remarks (1)

In this talk, we considered preservation of DP and EP by two schemata *MP* and *LPO* in the setting of intermediate predicate logics.

	H-DP,H-EP	PD-EP
$\mathbf{H}_*$	YES	YES
$\mathbf{H}_* + MP$, $\mathbf{H}_* + LPO$	YES	NO

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In this talk, we considered preservation of DP and EP by two schemata *MP* and *LPO* in the setting of intermediate predicate logics.

	H-DP,H-EP	PD-EP
$\mathbf{H}_*$	YES	YES
$\mathbf{H}_* + MP$, $\mathbf{H}_* + LPO$	YES	NO
$\mathbf{H}_* + WLPO$, $\mathbf{H}_* + LLPO$	?	YES
$\mathbf{H}_* + CD$	YES	NO
$\mathbf{H}_* + WEM$	NO	YES

WLPO: $\forall x(p(x) \vee \neg p(x)) \to \neg\exists x p(x) \vee \neg\neg\exists x p(x)$,
LLPO: $\{\forall x(p(x) \vee \neg p(x)) \wedge \forall x(q(x) \vee \neg q(x)) \wedge \neg(\exists x p(x) \wedge \exists x q(x))\}$
$\to \neg\exists x p(x) \vee \neg\exists x q(x)$,
CD: $\forall x(p(x) \vee q) \to \forall x p(x) \vee q$, ($x$ is not free in q)
WEM: $\neg p \vee \neg\neg p$,

Background Story: Ono's Problem P52
Relations betwen DP and EP in Intermediate Logics

Relations?

In intermediate predicate logics: DP $\Rightarrow$ EP? EP $\Rightarrow$ DP?

- (Nakamura 1983) There exists an intermediate logic having DP but lacking EP. I.e., DP $\not\Rightarrow$ EP.
- EP $\Rightarrow$ DP? in intermediate logics
  Ono's Problem P52 (1987) (cf. Umezawa(1980), Minari(1983))

Background Story: Ono's Problem P52
Relations betwen DP and EP in Intermediate Logics

Proposition

In intermediate predicate logics, EP and DP are independent. I.e.,

- (Nakamura 1983) There exists an intermediate logic having DP but lacking EP. I.e., DP $\not\Rightarrow$ EP.
- (S. 2013-15) There exists an intermediate logic having EP but lacking DP. I.e., EP $\not\Rightarrow$ DP.

Theorem (S.2013-15)

If $\mathbf{L}$ is closed under the rule:

$$\frac{A \vee (p(x) \rightarrow p(y))}{A} \quad (ZR)$$

where x, y and p are distinct and do not occur in A.

Then, EP of $\mathbf{L}$ implies DP of $\mathbf{L}$.

Concluding Remarks (2)

- Do $\mathbf{H}_* + WLPO$ and $\mathbf{H}_* + LLPO$ have H-DP and/or H-EP?
- H-DP $\Leftrightarrow$ DP? H-EP $\Leftrightarrow$ EP?
  This problem is known as Ono's problem P54.

  Remark: In intermediate propositional logic, we have: H-DP $\Leftrightarrow$ DP.
- There must be waiting us other axiom schemata arising from constructive theories which are interesting from the viewpoint of intermediate logics!
- There must be waiting us other phenomena in intermediate logics which are interesting from the view point of constructive theories!

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