# Geometry and Effective Dimension

Stephen Binns
Qatar University
CTFM

September 6, 2015

# Effective Dimensions

## Definition

Effective Hausdorff dimension:

$$\dim_H(X) = \liminf_n \frac{C(X \upharpoonright n)}{n}.$$

Effective packing dimension:

$$\dim_p(X) = \limsup_n \frac{C(X \upharpoonright n)}{n}.$$

If $\dim_H X = \dim_p X$ we refer to $X$ as <u>regular</u> and write $\dim X$.
$A$ and $B$ are <u>mutually regular</u> if both are regular and

$$\lim_n \frac{C(A \upharpoonright n, B \upharpoonright n)}{n} \text{ exists.}$$

# Effective Dimensions

**Definition**

Effective Hausdorff dimension:

$$\dim_H(X) = \liminf_n \frac{C(X \upharpoonright n)}{n}.$$

Effective packing dimension:

$$\dim_p(X) = \limsup_n \frac{C(X \upharpoonright n)}{n}.$$

If $\dim_H X = \dim_p X$ we refer to $X$ as regular and write $\dim X$. $A$ and $B$ are mutually regular if both are regular and

$$\lim_n \frac{C(A \upharpoonright n, B \upharpoonright n)}{n} \text{ exists.}$$

# Effective Dimensions

**Definition**

Effective Hausdorff dimension:

$$\dim_H(X) = \liminf_n \frac{C(X \upharpoonright n)}{n}.$$

Effective packing dimension:

$$\dim_p(X) = \limsup_n \frac{C(X \upharpoonright n)}{n}.$$

If $\dim_H X = \dim_p X$ we refer to $X$ as <u>regular</u> and write $\dim X$.

$A$ and $B$ are <u>mutually regular</u> if both are regular and

$$\lim_n \frac{C(A \upharpoonright n, B \upharpoonright n)}{n} \text{ exists.}$$

# Effective Dimensions

## Definition

Effective Hausdorff dimension:

$$\dim_H(X) = \liminf_n \frac{C(X \upharpoonright n)}{n}.$$

Effective packing dimension:

$$\dim_p(X) = \limsup_n \frac{C(X \upharpoonright n)}{n}.$$

If $\dim_H X = \dim_p X$ we refer to $X$ as regular and write $\dim X$.
$A$ and $B$ are mutually regular if both are regular and

$$\lim_n \frac{C(A \upharpoonright n, B \upharpoonright n)}{n} \text{ exists.}$$

# A directed premetric

**Definition (The $d$-metric)**

$$d(X \to Y) := \limsup_n \frac{C(Y \upharpoonright n \mid X \upharpoonright n)}{n}.$$

$$d(X, Y) := \max\{d(X \to Y), d(Y \to X)\}.$$

$$X \sim_d Y := d(X, Y) = 0.$$

**Theorem**

*$d$ is a metric on the $d$-equivalence classes.*

# A directed premetric

**Definition (The $d$-metric)**

$$d(X \to Y) := \limsup_n \frac{C(Y \upharpoonright n \mid X \upharpoonright n)}{n}.$$

$$d(X, Y) := \max\{d(X \to Y), d(Y \to X)\}.$$

$$X \sim_d Y := d(X, Y) = 0.$$

**Theorem**

*$d$ is a metric on the $d$-equivalence classes.*

# A directed premetric

**Definition (The $d$-metric)**

$$d(X \to Y) := \limsup_n \frac{C(Y \upharpoonright n \mid X \upharpoonright n)}{n}.$$

$$d(X, Y) := \max\{d(X \to Y), d(Y \to X)\}.$$

$$X \sim_d Y := d(X, Y) = 0.$$

**Theorem**

*$d$ is a metric on the $d$-equivalence classes.*

# A directed premetric

**Definition (The *d*-metric)**

$$d(X \to Y) := \limsup_n \frac{C(Y\restriction n \,|\, X\restriction n)}{n}.$$

$$d(X, Y) := \max\{d(X \to Y), d(Y \to X)\}.$$

$$X \sim_d Y := d(X, Y) = 0.$$

**Theorem**

*$d$ is a metric on the $d$-equivalence classes.*

## Some Results:

- $d$ induces a path-connected topology on the set of regular reals.
- The set of regular reals is topologically complete under $d$.
- The set of regular reals contains no compact neighbourhoods under $d$.
- If $\dim X = \dim Y$, then $d(X \to Y) = d(Y \to X)$.

## Some Results:

- $d$ induces a path-connected topology on the set of regular reals.
- The set of regular reals is topologically complete under $d$.
- The set of regular reals contains no compact neighbourhoods under $d$.
- If $\dim X = \dim Y$, then $d(X \to Y) = d(Y \to X)$.

# Some Results:

- $d$ induces a path-connected topology on the set of regular reals.
- The set of regular reals is topologically complete under $d$.
- The set of regular reals contains no compact neighbourhoods under $d$.
- If $\dim X = \dim Y$, then $d(X \to Y) = d(Y \to X)$.

# Some Results:

- $d$ induces a path-connected topology on the set of regular reals.
- The set of regular reals is topologically complete under $d$.
- The set of regular reals contains no compact neighbourhoods under $d$.
- If $\dim X = \dim Y$, then $d(X \to Y) = d(Y \to X)$.

## Some Results:

- $d$ induces a path-connected topology on the set of regular reals.
- The set of regular reals is topologically complete under $d$.
- The set of regular reals contains no compact neighbourhoods under $d$.
- If $\dim X = \dim Y$, then $d(X \to Y) = d(Y \to X)$.

# Dilutions

**Definition**

Let $\alpha \in [0,1]$ and $X = x_0 x_1 x_2 \ldots$

$$\alpha X := x_0 x_1 \ldots x_{i_1} 000 \ldots 0 x_{i_1+1} x_{i_1+2} \ldots x_{i_2} 000 \ldots 0 x_{i_2+1} \ldots$$

Where $|x_{i_j+1} \ldots x_{i_{j+1}} 000 \ldots 0| = j+1$, and

$$|x_{i_j+1} \ldots x_{i_{j+1}}| = \lfloor \alpha(j+1) \rfloor.$$

$\alpha X$ is the $\alpha$-dilution of $X$.

# Properties of Dilution

1. $0X = 0000\cdots \sim_d$ any real of dimension 0,
2. If $X$ is regular, then $\alpha X$ is regular,
3. $\dim(\alpha X) = \alpha \dim(X)$,
4. $\alpha(\beta X) \simeq_d (\alpha\beta)X$,
5. $d(\alpha X \to \alpha Y) = \alpha d(X \to Y)$,
6. $d(\alpha X \to \beta X) = \begin{cases} 0 & \text{if } \alpha \geqslant \beta, \\ (\beta - \alpha)\dim X & \text{otherwise,} \end{cases}$
7. The map $\alpha \mapsto \alpha X$ is a continuous map from $[0,1]$ into $2^{\mathbb{N}}$ under the $d$-topology.

# Properties of Dilution

1. $0X = 0000\cdots \sim_d$ any real of dimension 0,
2. If $X$ is regular, then $\alpha X$ is regular,
3. $\dim(\alpha X) = \alpha \dim(X)$,
4. $\alpha(\beta X) \simeq_d (\alpha\beta)X$,
5. $d(\alpha X \to \alpha Y) = \alpha d(X \to Y)$,
6. $d(\alpha X \to \beta X) = \begin{cases} 0 & \text{if } \alpha \geqslant \beta, \\ (\beta - \alpha)\dim X & \text{otherwise,} \end{cases}$
7. The map $\alpha \mapsto \alpha X$ is a continuous map from $[0, 1]$ into $2^{\mathbb{N}}$ under the $d$-topology.

# Properties of Dilution

1. $0X = 0000\cdots \sim_d$ any real of dimension 0,
2. If $X$ is regular, then $\alpha X$ is regular,
3. $\dim(\alpha X) = \alpha \dim(X)$,
4. $\alpha(\beta X) \simeq_d (\alpha\beta)X$,
5. $d(\alpha X \to \alpha Y) = \alpha d(X \to Y)$,
6. $d(\alpha X \to \beta X) = \begin{cases} 0 & \text{if } \alpha \geqslant \beta, \\ (\beta - \alpha)\dim X & \text{otherwise,} \end{cases}$
7. The map $\alpha \mapsto \alpha X$ is a continuous map from $[0, 1]$ into $2^{\mathbb{N}}$ under the $d$-topology.

# Properties of Dilution

1. $0X = 0000\cdots \sim_d$ any real of dimension 0,
2. If $X$ is regular, then $\alpha X$ is regular,
3. $\dim(\alpha X) = \alpha \dim(X)$,
4. $\alpha(\beta X) \simeq_d (\alpha\beta)X$,
5. $d(\alpha X \to \alpha Y) = \alpha d(X \to Y)$,
6. $d(\alpha X \to \beta X) = \begin{cases} 0 & \text{if } \alpha \geqslant \beta, \\ (\beta - \alpha)\dim X & \text{otherwise,} \end{cases}$
7. The map $\alpha \mapsto \alpha X$ is a continuous map from $[0, 1]$ into $2^{\mathbb{N}}$ under the $d$-topology.

# Properties of Dilution

1. $0X = 0000\cdots \sim_d$ any real of dimension 0,
2. If $X$ is regular, then $\alpha X$ is regular,
3. $\dim(\alpha X) = \alpha \dim(X)$,
4. $\alpha(\beta X) \simeq_d (\alpha\beta)X$,
5. $d(\alpha X \to \alpha Y) = \alpha d(X \to Y)$,
6. $d(\alpha X \to \beta X) = \begin{cases} 0 & \text{if } \alpha \geqslant \beta, \\ (\beta - \alpha)\dim X & \text{otherwise,} \end{cases}$
7. The map $\alpha \mapsto \alpha X$ is a continuous map from $[0, 1]$ into $2^{\mathbb{N}}$ under the $d$-topology.

# Properties of Dilution

1. $0X = 0000\cdots \sim_d$ any real of dimension 0,
2. If $X$ is regular, then $\alpha X$ is regular,
3. $\dim(\alpha X) = \alpha \dim(X)$,
4. $\alpha(\beta X) \simeq_d (\alpha\beta)X$,
5. $d(\alpha X \to \alpha Y) = \alpha d(X \to Y)$,
6. $d(\alpha X \to \beta X) = \begin{cases} 0 & \text{if } \alpha \geqslant \beta, \\ (\beta - \alpha)\dim X & \text{otherwise,} \end{cases}$
7. The map $\alpha \mapsto \alpha X$ is a continuous map from $[0,1]$ into $2^{\mathbb{N}}$ under the $d$-topology.

# Properties of Dilution

1. $0X = 0000 \cdots \sim_d$ any real of dimension 0,
2. If $X$ is regular, then $\alpha X$ is regular,
3. $\dim(\alpha X) = \alpha \dim(X)$,
4. $\alpha(\beta X) \simeq_d (\alpha\beta)X$,
5. $d(\alpha X \to \alpha Y) = \alpha d(X \to Y)$,
6. $d(\alpha X \to \beta X) = \begin{cases} 0 & \text{if } \alpha \geqslant \beta, \\ (\beta - \alpha) \dim X & \text{otherwise,} \end{cases}$
7. The map $\alpha \mapsto \alpha X$ is a continuous map from $[0, 1]$ into $2^{\mathbb{N}}$ under the $d$-topology.

# Compression

**Theorem (Binns, Nicholson)**

*For every regular $X$ of dimension $\alpha$ there is a unique (up to $d$-equivalence) $Y$ of dimension 1, such that $X \simeq_d \alpha Y$.*

**Proof.**

We let $X = \tau_1\tau_2\tau_3\dots$ where $|\tau_i| = i$. Then we define

- $\gamma_1 = \tau_1$
- $\gamma_{i+1} = (\tau_{i+1} \mid \tau_1, \tau_2, \dots \tau_i)^*$,

where the notation $(\tau_{i+1} \mid \tau_1, \tau_2, \dots \tau_i)^*$ denotes a program of minimal length that outputs $\tau_{i+1}$ given $\tau_1, \tau_2, \tau_3, \dots \tau_i$ as input. Now we let $Y = \gamma_1\gamma_2\gamma_3\dots$. □

# Compression

**Theorem (Binns, Nicholson)**

*For every regular $X$ of dimension $\alpha$ there is a unique (up to $d$-equivalence) $Y$ of dimension 1, such that $X \simeq_d \alpha Y$.*

**Proof.**

We let $X = \tau_1\tau_2\tau_3\dots$ where $|\tau_i| = i$. Then we define

- $\gamma_1 = \tau_1$
- $\gamma_{i+1} = (\tau_{i+1} \,|\, \tau_1, \tau_2, \dots \tau_i)^*$,

where the notation $(\tau_{i+1} \,|\, \tau_1, \tau_2, \dots \tau_i)^*$ denotes a program of minimal length that outputs $\tau_{i+1}$ given $\tau_1, \tau_2, \tau_3, \dots \tau_i$ as input. Now we let $Y = \gamma_1\gamma_2\gamma_3\dots$. $\square$

## Proof cont.

**Proof (very basic sketch).**

Let $Y_i = \gamma_1\gamma_2\gamma_3\ldots\gamma_i$, and $X_i = \tau_1\tau_2\tau_3\ldots\tau_i$
We show that:

- $|X_i| = \mathcal{O}(i^2)$
- $|Y_i| = C(X_i) \pm \mathcal{O}(i \log i)$.
- $d(\alpha Y \to X) = 0$
- $\dim Y = 1$
- As $\dim(\alpha Y) = \alpha = \dim(X)$, this implies $d(X \to \alpha Y) = 0$ and $\alpha Y \sim_d X$.

# Proof cont.

**Proof (very basic sketch).**

Let $Y_i = \gamma_1\gamma_2\gamma_3\ldots\gamma_i$, and $X_i = \tau_1\tau_2\tau_3\ldots\tau_i$
We show that:

- $|X_i| = \mathcal{O}(i^2)$
- $|Y_i| = C(X_i) \pm \mathcal{O}(i \log i)$.
- $d(\alpha Y \to X) = 0$
- $\dim Y = 1$
- As $\dim(\alpha Y) = \alpha = \dim(X)$, this implies $d(X \to \alpha Y) = 0$ and $\alpha Y \sim_d X$.

# Proof cont.

**Proof (very basic sketch).**

Let $Y_i = \gamma_1\gamma_2\gamma_3 \ldots \gamma_i$, and $X_i = \tau_1\tau_2\tau_3 \ldots \tau_i$
We show that:

- $|X_i| = \mathcal{O}(i^2)$
- $|Y_i| = C(X_i) \pm \mathcal{O}(i \log i)$.
- $d(\alpha Y \to X) = 0$
- $\dim Y = 1$
- As $\dim(\alpha Y) = \alpha = \dim(X)$, this implies $d(X \to \alpha Y) = 0$ and $\alpha Y \sim_d X$.

$\square$

# Proof cont.

**Proof (very basic sketch).**

Let $Y_i = \gamma_1\gamma_2\gamma_3 \ldots \gamma_i$, and $X_i = \tau_1\tau_2\tau_3 \ldots \tau_i$
We show that:

- $|X_i| = \mathcal{O}(i^2)$
- $|Y_i| = C(X_i) \pm \mathcal{O}(i \log i)$.
- $d(\alpha Y \to X) = 0$
- $\dim Y = 1$
- As $\dim(\alpha Y) = \alpha = \dim(X)$, this implies $d(X \to \alpha Y) = 0$ and $\alpha Y \sim_d X$.

□

# Proof cont.

**Proof (very basic sketch).**

Let $Y_i = \gamma_1\gamma_2\gamma_3 \dots \gamma_i$, and $X_i = \tau_1\tau_2\tau_3 \dots \tau_i$
We show that:

- $|X_i| = \mathcal{O}(i^2)$
- $|Y_i| = C(X_i) \pm \mathcal{O}(i \log i)$.
- $d(\alpha Y \to X) = 0$
- $\dim Y = 1$
- As $\dim(\alpha Y) = \alpha = \dim(X)$, this implies $d(X \to \alpha Y) = 0$ and $\alpha Y \sim_d X$.

$\square$

# $\dim Y = 1$ (Proof sketch)

**Proof.**

$$
\begin{aligned}
C(\gamma_{i+1} \mid Y_i) + C(Y_i \mid X_i) &\geqslant C(\tau_{i+1} \mid X_i) - o(i^2) \\
&= |\gamma_{i+1}| - o(i^2).
\end{aligned}
$$

But

$$
\begin{aligned}
C(Y_i \mid X_i) &= C(Y_i, X_i) - C(X_i) \pm o(i^2) \\
&= C(Y_i) - C(X_i) \pm o(i^2).
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad C(\gamma_{i+1} \mid Y_i) + C(Y_i) - C(X_i) &\geqslant |\gamma_{i+1}| \pm o(i^2) \\
\Rightarrow \quad C(Y_{i+1}) &\geqslant |\gamma_{i+1}| + C(X_i) \pm o(i^2) \\
&= |Y_{i+1}| \pm o(i^2)
\end{aligned}
$$

$\square$

# $\dim Y = 1$ (Proof sketch)

**Proof.**

$$
\begin{aligned}
C(\gamma_{i+1} \mid Y_i) + C(Y_i \mid X_i) &\geqslant C(\tau_{i+1} \mid X_i) - o(i^2) \\
&= |\gamma_{i+1}| - o(i^2).
\end{aligned}
$$

But
$$
\begin{aligned}
C(Y_i \mid X_i) &= C(Y_i, X_i) - C(X_i) \pm o(i^2) \\
&= C(Y_i) - C(X_i) \pm o(i^2).
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad C(\gamma_{i+1} \mid Y_i) + C(Y_i) - C(X_i) &\geqslant |\gamma_{i+1}| \pm o(i^2) \\
\Rightarrow \quad C(Y_{i+1}) &\geqslant |\gamma_{i+1}| + C(X_i) \pm o(i^2) \\
&= |Y_{i+1}| \pm o(i^2)
\end{aligned}
$$

$\square$

# $\dim Y = 1$ (Proof sketch)

**Proof.**

$$\begin{aligned} C(\gamma_{i+1} \mid Y_i) + C(Y_i \mid X_i) &\geqslant C(\tau_{i+1} \mid X_i) - o(i^2) \\ &= |\gamma_{i+1}| - o(i^2). \end{aligned}$$

But

$$\begin{aligned} C(Y_i \mid X_i) &= C(Y_i, X_i) - C(X_i) \pm o(i^2) \\ &= C(Y_i) - C(X_i) \pm o(i^2). \end{aligned}$$

$$\begin{aligned} \therefore \quad C(\gamma_{i+1} \mid Y_i) + C(Y_i) - C(X_i) &\geqslant |\gamma_{i+1}| \pm o(i^2) \\ \Rightarrow \quad C(Y_{i+1}) &\geqslant |\gamma_{i+1}| + C(X_i) \pm o(i^2) \\ &= |Y_{i+1}| \pm o(i^2) \end{aligned}$$

$\square$

## $\dim Y = 1$ (Proof sketch)

**Proof.**

$$\begin{aligned} C(\gamma_{i+1} \mid Y_i) + C(Y_i \mid X_i) &\geqslant C(\tau_{i+1} \mid X_i) - o(i^2) \\ &= |\gamma_{i+1}| - o(i^2). \end{aligned}$$

But
$$\begin{aligned} C(Y_i \mid X_i) &= C(Y_i, X_i) - C(X_i) \pm o(i^2) \\ &= C(Y_i) - C(X_i) \pm o(i^2). \end{aligned}$$

$$\begin{aligned} \therefore \quad C(\gamma_{i+1} \mid Y_i) + C(Y_i) - C(X_i) &\geqslant |\gamma_{i+1}| \pm o(i^2) \\ \Rightarrow \quad C(Y_{i+1}) &\geqslant |\gamma_{i+1}| + C(X_i) \pm o(i^2) \\ &= |Y_{i+1}| \pm o(i^2) \end{aligned}$$

$\square$

# $\dim Y = 1$ (Proof sketch)

**Proof.**

$$C(\gamma_{i+1} \mid Y_i) + C(Y_i \mid X_i) \geqslant C(\tau_{i+1} \mid X_i) - o(i^2) = |\gamma_{i+1}| - o(i^2).$$

But

$$C(Y_i \mid X_i) = C(Y_i, X_i) - C(X_i) \pm o(i^2) = C(Y_i) - C(X_i) \pm o(i^2).$$

$$\therefore \quad C(\gamma_{i+1} \mid Y_i) + C(Y_i) - C(X_i) \geqslant |\gamma_{i+1}| \pm o(i^2)$$

$$\Rightarrow \quad C(Y_{i+1}) \geqslant |\gamma_{i+1}| + C(X_i) \pm o(i^2) = |Y_{i+1}| \pm o(i^2)$$

□

# $\dim Y = 1$ (Proof sketch)

**Proof.**

$$\begin{aligned} C(\gamma_{i+1} \mid Y_i) + C(Y_i \mid X_i) &\geqslant C(\tau_{i+1} \mid X_i) - o(i^2) \\ &= |\gamma_{i+1}| - o(i^2). \end{aligned}$$

But

$$\begin{aligned} C(Y_i \mid X_i) &= C(Y_i, X_i) - C(X_i) \pm o(i^2) \\ &= C(Y_i) - C(X_i) \pm o(i^2). \end{aligned}$$

$$\begin{aligned} \therefore \quad C(\gamma_{i+1} \mid Y_i) + C(Y_i) - C(X_i) &\geqslant |\gamma_{i+1}| \pm o(i^2) \\ \Rightarrow \quad C(Y_{i+1}) &\geqslant |\gamma_{i+1}| + C(X_i) \pm o(i^2) \\ &= |Y_{i+1}| \pm o(i^2) \end{aligned}$$

$\square$

# $\dim Y = 1$ (Proof sketch)

**Proof.**

$$
\begin{aligned}
C(\gamma_{i+1} \mid Y_i) + C(Y_i \mid X_i) &\geqslant C(\tau_{i+1} \mid X_i) - o(i^2) \\
&= |\gamma_{i+1}| - o(i^2).
\end{aligned}
$$

But

$$
\begin{aligned}
C(Y_i \mid X_i) &= C(Y_i, X_i) - C(X_i) \pm o(i^2) \\
&= C(Y_i) - C(X_i) \pm o(i^2).
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad C(\gamma_{i+1} \mid Y_i) + C(Y_i) - C(X_i) &\geqslant |\gamma_{i+1}| \pm o(i^2) \\
\Rightarrow \quad C(Y_{i+1}) &\geqslant |\gamma_{i+1}| + C(X_i) \pm o(i^2) \\
&= |Y_{i+1}| \pm o(i^2)
\end{aligned}
$$

$\square$

# Randomness and Complexity

**Theorem**

*Given any regular real $X$ of dimension $\alpha > 0$, there is a 1-random real $R$ such that $X \sim_d \alpha R$.*

**Lemma (Kučera, Gác & Merkle, Mihailović)**

*There is a partial computable functional $\Phi$ on $2^{\mathbb{N}}$ with the properties:*

1. *For every $Y \in 2^{\mathbb{N}}$ there is a Martin-Löf random $R$ such that $\Phi^R = Y$.*
2. *There is a computable function $g$ such that $g(n)$ bounds the use of $R$ in calculating $Y \upharpoonright n$.*
3. $\limsup \frac{g(n)}{n} \leqslant 1$.

# Randomness and Complexity

**Theorem**

*Given any regular real $X$ of dimension $\alpha > 0$, there is a 1-random real $R$ such that $X \sim_d \alpha R$.*

**Lemma (Kučera, Gác & Merkle, Mihailović)**

*There is a partial computable functional $\Phi$ on $2^{\mathbb{N}}$ with the properties:*

1. *For every $Y \in 2^{\mathbb{N}}$ there is a Martin-Löf random $R$ such that $\Phi^R = Y$.*
2. *There is a computable function $g$ such that $g(n)$ bounds the use of $R$ in calculating $Y \upharpoonright n$.*
3. $\limsup \frac{g(n)}{n} \leqslant 1$.

# Proof cont.

**Proof.**

Given $X$ of dimension $\alpha$, construct $Y$ of dimension 1 such that $X \sim_d \alpha Y$. Now take $R$ as in Lemma for $Y$. To describe $Y \upharpoonright n$ given $R \upharpoonright n$ we need only an extra $g(n) - n$ bits. Thus

$$\begin{aligned} d(R \to Y) &= \limsup_n \frac{C(Y \upharpoonright n \mid R \upharpoonright n)}{n} \\ &\leqslant \limsup_n \frac{g(n) - n + \mathcal{O}(1)}{n} \\ &\leqslant \limsup_n \frac{g(n)}{n} - 1 \\ &\leqslant 0 \end{aligned}$$

As $\dim R = 1 = \dim Y$, we have also that $d(Y \to R) = 0$ and $Y \sim_d R$. Finally $\alpha R \sim_d \alpha Y \sim_d X$ as required. ∎

# Proof cont.

**Proof.**

Given $X$ of dimension $\alpha$, construct $Y$ of dimension 1 such that $X \sim_d \alpha Y$. Now take $R$ as in Lemma for $Y$. To describe $Y \upharpoonright n$ given $R \upharpoonright n$ we need only an extra $g(n) - n$ bits. Thus

$$\begin{aligned} d(R \to Y) &= \limsup_n \frac{C(Y \upharpoonright n \mid R \upharpoonright n)}{n} \\ &\leqslant \limsup_n \frac{g(n) - n + \mathcal{O}(1)}{n} \\ &\leqslant \limsup_n \frac{g(n)}{n} - 1 \\ &\leqslant 0 \end{aligned}$$

As $\dim R = 1 = \dim Y$, we have also that $d(Y \to R) = 0$ and $Y \sim_d R$. Finally $\alpha R \sim_d \alpha Y \sim_d X$ as required. $\square$

# Proof cont.

**Proof.**

Given $X$ of dimension $\alpha$, construct $Y$ of dimension 1 such that $X \sim_d \alpha Y$. Now take $R$ as in Lemma for $Y$. To describe $Y \upharpoonright n$ given $R \upharpoonright n$ we need only an extra $g(n) - n$ bits. Thus

$$\begin{aligned} d(R \to Y) &= \limsup_n \frac{C(Y \upharpoonright n \mid R \upharpoonright n)}{n} \\ &\leqslant \limsup_n \frac{g(n) - n + \mathcal{O}(1)}{n} \\ &\leqslant \limsup_n \frac{g(n)}{n} - 1 \\ &\leqslant 0 \end{aligned}$$

As $\dim R = 1 = \dim Y$, we have also that $d(Y \to R) = 0$ and $Y \sim_d R$. Finally $\alpha R \sim_d \alpha Y \sim_d X$ as required. $\square$

# Proof cont.

**Proof.**

Given $X$ of dimension $\alpha$, construct $Y$ of dimension 1 such that $X \sim_d \alpha Y$. Now take $R$ as in Lemma for $Y$. To describe $Y \restriction n$ given $R \restriction n$ we need only an extra $g(n) - n$ bits. Thus

$$\begin{aligned}
d(R \to Y) &= \limsup_n \frac{C(Y \restriction n \mid R \restriction n)}{n} \\
&\leqslant \limsup_n \frac{g(n) - n + \mathcal{O}(1)}{n} \\
&\leqslant \limsup_n \frac{g(n)}{n} - 1 \\
&\leqslant 0
\end{aligned}$$

As $\dim R = 1 = \dim Y$, we have also that $d(Y \to R) = 0$ and $Y \sim_d R$. Finally $\alpha R \sim_d \alpha Y \sim_d X$ as required. $\square$

# Convex combinations

We can pad a real $X$ with bits from another real $Y$:
Let $r \in [0,1]$ and $X = x_0 x_1 x_2 \ldots$, $Y = y_0 y_1 y_2 \ldots$
Define

$$r[X, Y] = x_0 x_1 \ldots x_{i_1} y_0 y_1 \ldots y_{j_1} x_{i_1+1} x_{i_1+2} \ldots x_{i_2} y_{j_1+1} \ldots y_{j_2} x_{i_2+1} \ldots$$

Where $|x_{i_k+1} \ldots x_{i_{k+1}} y_{i_k+1} \ldots y_{i_{k+1}}| = k+1$, and

$$i_{k+1} = \lfloor r(k+1) \rfloor.$$

**Definition**

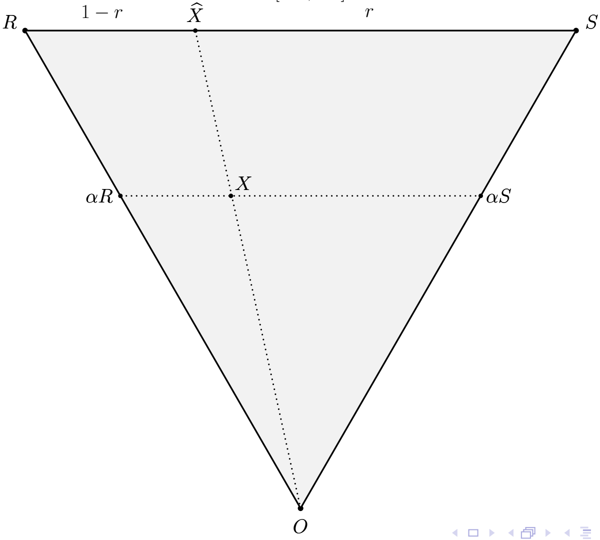
The <u>convex hull of $X$ and $Y$</u> is

$$\mathcal{H}(X, Y) = \{r[\alpha X, \alpha Y] : r, \alpha \in [0,1]\}$$

closed under $d$-equivalence.

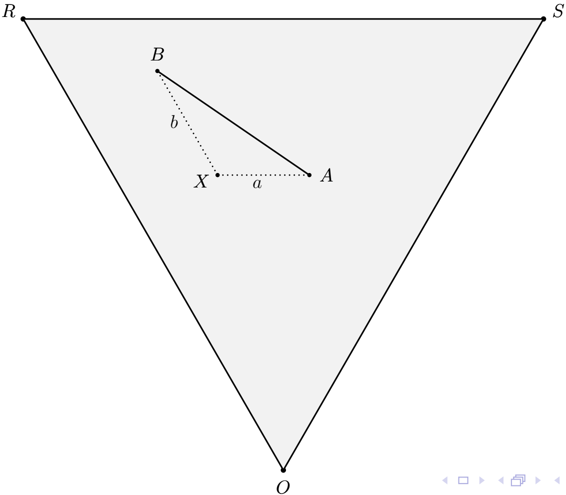
# The convex hull of mutually random reals $R$ and $S$

$X = r[\alpha R, \alpha S]$.



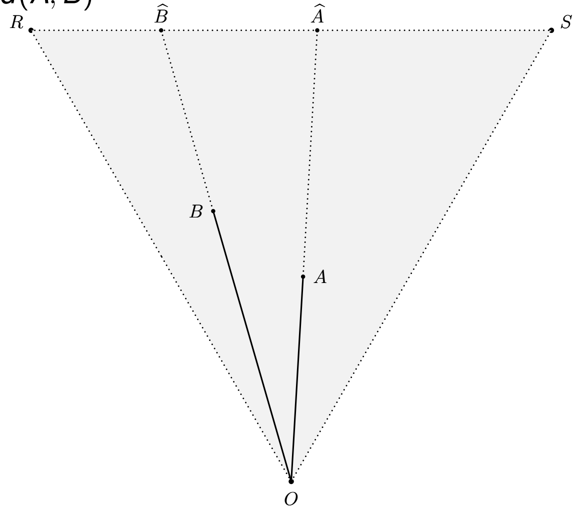
# The distance function

$d(B \to X) = 0,\ d(X \to B) = b,\ d(X \to A) = d(A \to X) = a,$
$d(B \to A) = a,\ d(A \to B) = a + b.$



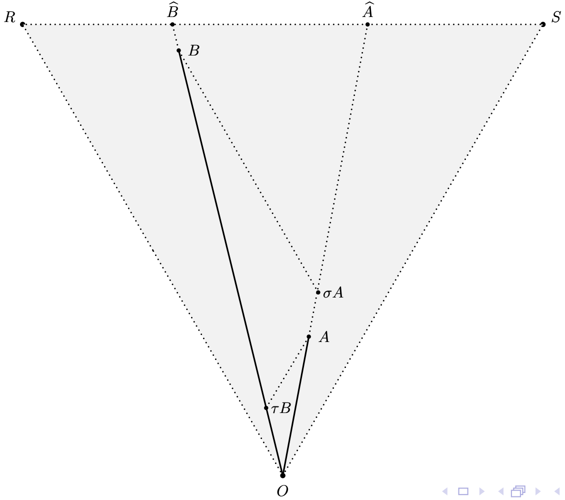
# Angles

$$\angle AB = d(\widehat{A}, \widehat{B})$$



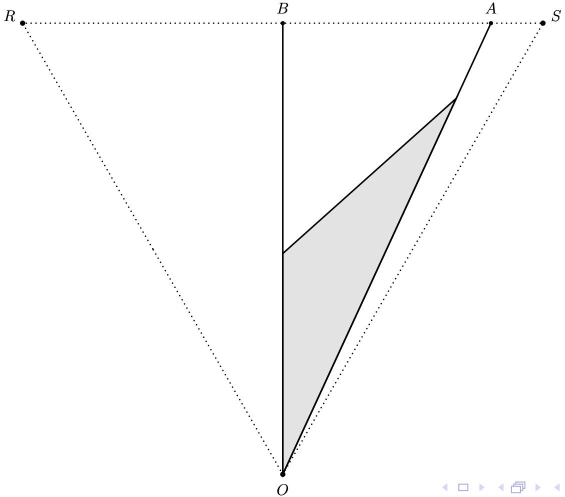
## **Projections:** $\mathrm{Proj}_X Y := \sup\{\alpha : d(Y \to \alpha X) = 0\}$

$\tau = \mathrm{Proj}_B A \qquad \sigma = \mathrm{Proj}_A B.$



## Other hulls: $A, B \in \mathcal{H}(R, S)$.

$$B = \tfrac{1}{2}[R, S], \qquad A = \tfrac{1}{10}[R, S].$$



**Question**

What kind of geometry can $\mathcal{H}(A, B)$ exhibit?

The previous examples have planar hulls. They can be isometrically embedded in $\mathcal{H}(R, S)$.
Are all hulls planar? No.

# Other hulls: Arbitrary mutually regular $A$ and $B$

**Definition**

Let $A$ and $B$ be mutually regular reals of dimension 1. Then $A$ and $B$ form a coherent pair if

$$\angle AB = \frac{(1-\sigma)(1-\tau)}{1-\sigma\tau},$$

where $\sigma = \mathrm{Proj}_A B$ and $\tau = \mathrm{Proj}_B A$.

**Theorem**

*$\mathcal{H}(A, B)$ is planar if an only if $A$ and $B$ are coherent.*

**Theorem**

*Not all mutually regular reals are coherent.*

# Other hulls: Arbitrary mutually regular $A$ and $B$

**Definition**

Let $A$ and $B$ be mutually regular reals of dimensions $a$ and $b$ respectively. Then $A$ and $B$ form a coherent pair if

$$\angle AB = \frac{(b - a\sigma)(a - b\tau)}{ab(1 - \sigma\tau)},$$

where $\sigma = \mathrm{Proj}_A B$ and $\tau = \mathrm{Proj}_B A$.

**Theorem**

*$\mathcal{H}(A, B)$ is planar if an only if $A$ and $B$ are coherent.*

**Theorem**

*Not all mutually regular reals are coherent.*

# Not all $A$ $B$ are coherent

**Proof.**

Let $R = r_1 r_2 r_3 r_4 \dots r_n \dots$ be a random real. Let

$$A = r_0 r_2 r_4 \dots r_{2n} \dots$$

$$B = r_0 r_3 r_6 \dots r_{3n} \dots$$

Both $A$ and $B$ are random and so are dimension 1. But

- $\angle AB = d(B \to A) = 2/3$
- $\mathrm{Proj}_B A = 0$
- $\mathrm{Proj}_A B = 0$
- $2/3 \neq \frac{(1-0)(1-0)}{1-0} = 1.$

$\square$

# Other directions

- Given a coherent pair $A$ and $B$, does there exist a pair of mutually random reals $R, S$ such that

$$\mathcal{H}(A, B) \subseteq \mathcal{H}(R, S)?$$

- Given a mutually regular pair $A$, $B$ is there a curve of length $d(A, B)$ connecting $A$ and $B$?
- Given a mutually regular pair $A$ and $B$, does there exist a (unique?) $C$ of minimal dimension such that $d(B, C \to A) = 0$? Where

$$d(B, C \to A) := \limsup_n \frac{C(A \upharpoonright n \mid B \upharpoonright n \oplus C \upharpoonright n)}{n}.$$

# Other directions

- Given a coherent pair $A$ and $B$, does there exist a pair of mutually random reals $R, S$ such that

$$\mathcal{H}(A, B) \subseteq \mathcal{H}(R, S)?$$

- Given a mutually regular pair $A$, $B$ is there a curve of length $d(A, B)$ connecting $A$ and $B$?
- Given a mutually regular pair $A$ and $B$, does there exist a (unique?) $C$ of minimal dimension such that $d(B, C \to A) = 0$? Where

$$d(B, C \to A) := \limsup_n \frac{C(A\restriction n \mid B\restriction n \oplus C\restriction n)}{n}.$$

# Other directions

- Given a coherent pair $A$ and $B$, does there exist a pair of mutually random reals $R, S$ such that

$$\mathcal{H}(A, B) \subseteq \mathcal{H}(R, S)?$$

- Given a mutually regular pair $A$, $B$ is there a curve of length $d(A, B)$ connecting $A$ and $B$?
- Given a mutually regular pair $A$ and $B$, does there exist a (unique?) $C$ of minimal dimension such that $d(B, C \to A) = 0$? Where

$$d(B, C \to A) := \limsup_n \frac{C(A\restriction n \mid B\restriction n \oplus C\restriction n)}{n}.$$

# Other directions

- Given a coherent pair $A$ and $B$, does there exist a pair of mutually random reals $R, S$ such that

$$\mathcal{H}(A,B) \subseteq \mathcal{H}(R,S)?$$

- Given a mutually regular pair $A$, $B$ is there a curve of length $d(A,B)$ connecting $A$ and $B$?
- Given a mutually regular pair $A$ and $B$, does there exist a (unique?) $C$ of minimal dimension such that $d(B, C \to A) = 0$? Where

$$d(B, C \to A) := \limsup_n \frac{C(A \upharpoonright n \mid B \upharpoonright n \oplus C \upharpoonright n)}{n}.$$

# Arigatou gozaimasu

# Bibliography


Stephen Binns.
Relative Kolmogorov Complexity and Geometry.
Journal of Symbolic Logic, 76(4), 2011.

Stephen Binns.
Completeness, compactness, effective dimension.
Mathematical Logic Quarterly, 59(3):206–218, May 2013.

Stephen Binns and Marie Nicholson.
Compressibility and Kolmogorov complexity.
Notre Dame Journal of Formal Logic, 54(1):105–123, 2013.

H.D. Ebbinghaus, G.H. Müller, and G.E. Sacks, editors.
Recursion Theory Week.
Number 1141 in Lecture Notes in Mathematics.
Springer-Verlag, 1985.
IX + 418 pages.

# Bibliography cont.


P. Gács.
Every sequence is reducible to a random one.
Information and Control, 70:186–192, 1986.

Antonín Kučera.
Measure, $\Pi^0_1$ classes and complete extensions of PA.
In [4], pages 245–259, 1985.

Wolfgang Merkle, Nenad Mihailović.
On the construction of effectively random sets.
Journal of Symbolic Logic, Vol. 69, Issue 3, pages 862–878, 2004.