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Sequent calculi of quantum logic with strict implication

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- About quantum logic
- Sequent calculi for quantum logic with implication
- Labeled sequent for quantum logic and cut elimination

1 Quantum Logic

Quantum Logic is one of **Non-classical logic** which is based on proposition of quantum physics.

Quantum logic does not satisfies
the **Distributive law** $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$

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- Quantum Logic + **Distributive law** = Classical Logic
- Intuitionistic Logic + **Excluded middle** = Classical Logic

1.1 Semantics of quantum logic

· Orthomodular lattice
(Ortho lattice : **Minimal quantum logic**)

· Kripke model

Ortho lattice $(B, \sqsubseteq, \sqcap, \sqcup, ', \mathbf{1}, \mathbf{0})$

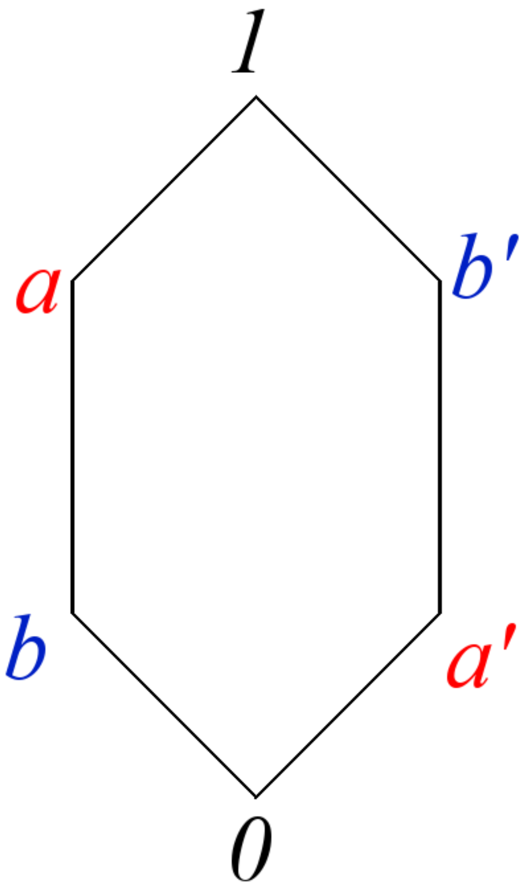
$\forall a \in B$,

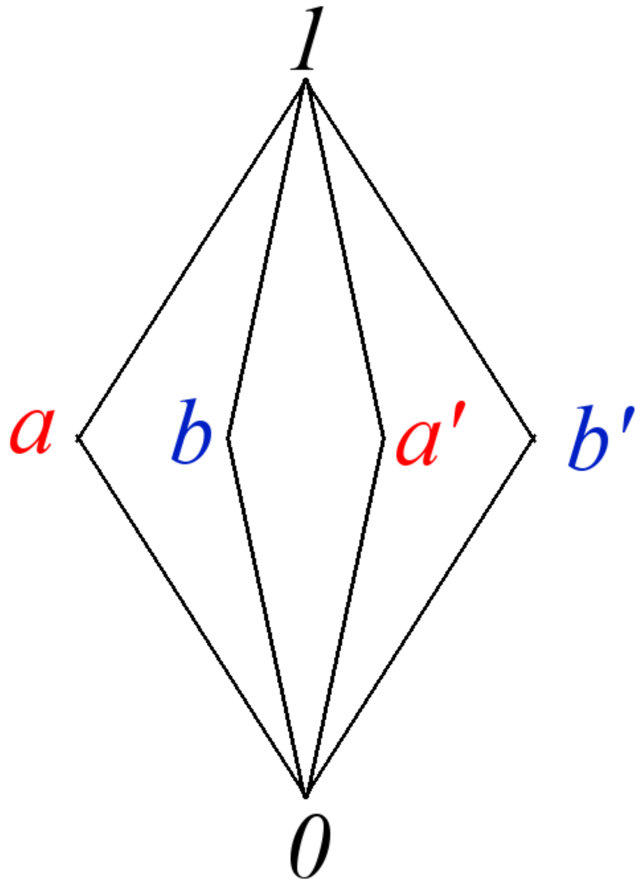
$$a'' = a$$

$$a \sqsubseteq b \quad \Rightarrow \quad b' \sqsubseteq a'$$

$$a \sqcap a' = \mathbf{0}, \;\; a \sqcup a' = \mathbf{1}$$

$$a \sqcup b = (a' \sqcap b')'$$

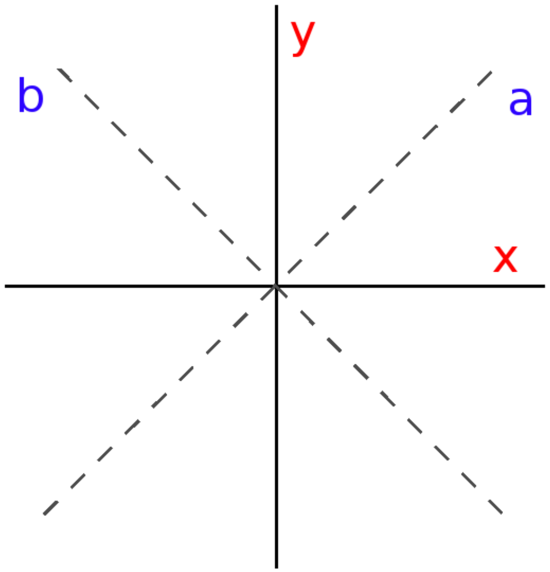


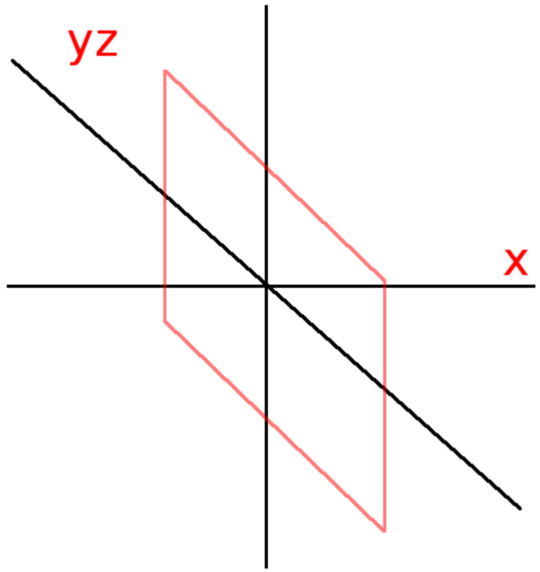


Distributive law is not satisfied in both of these lattice.

$a \sqcap (b \sqcup a') \neq (a \sqcap b) \sqcup (a \sqcap a')$

Quantum logic is based on closed subspace of Hilbert space.

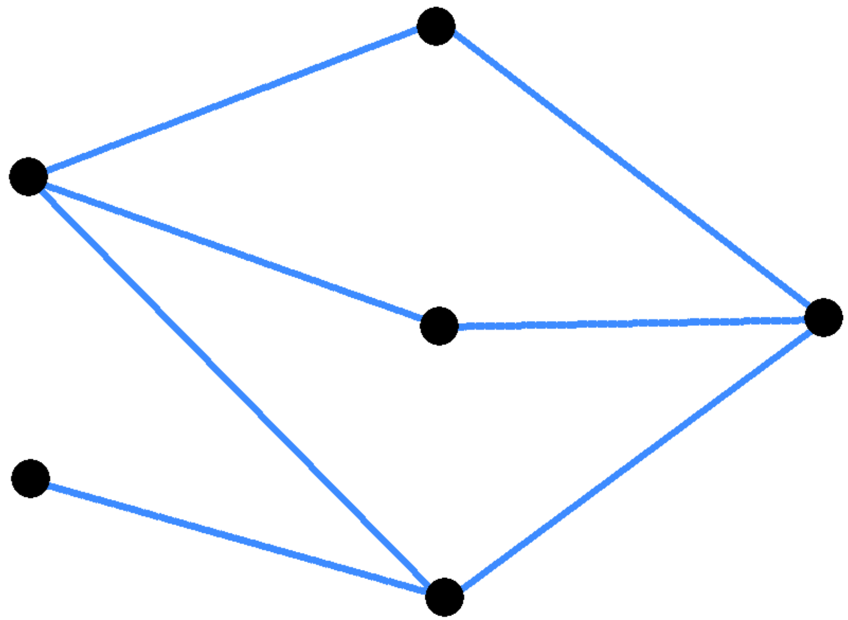


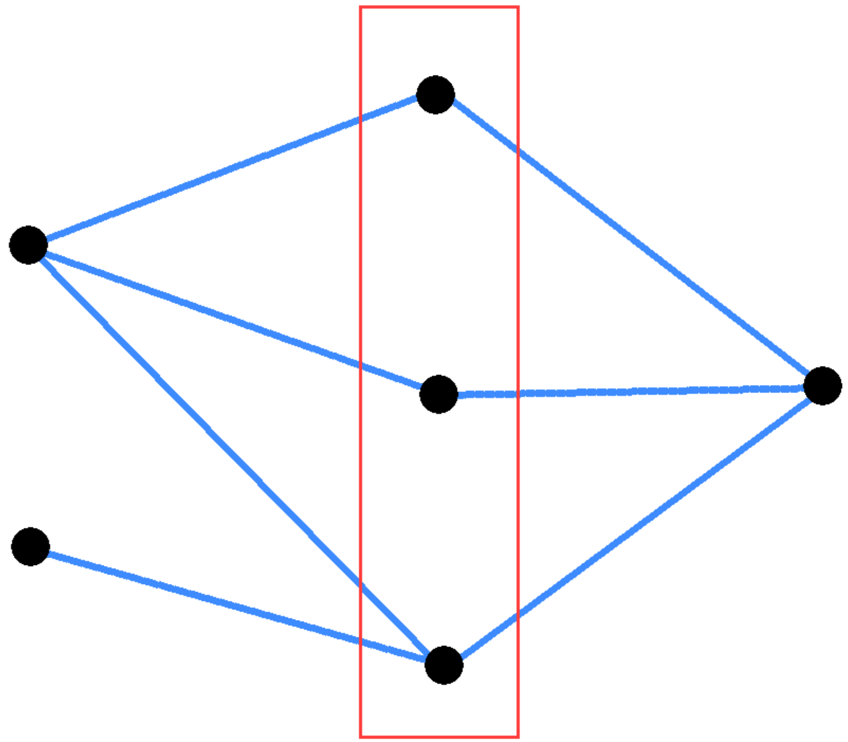


O-model is triple $(X, \perp, V)$

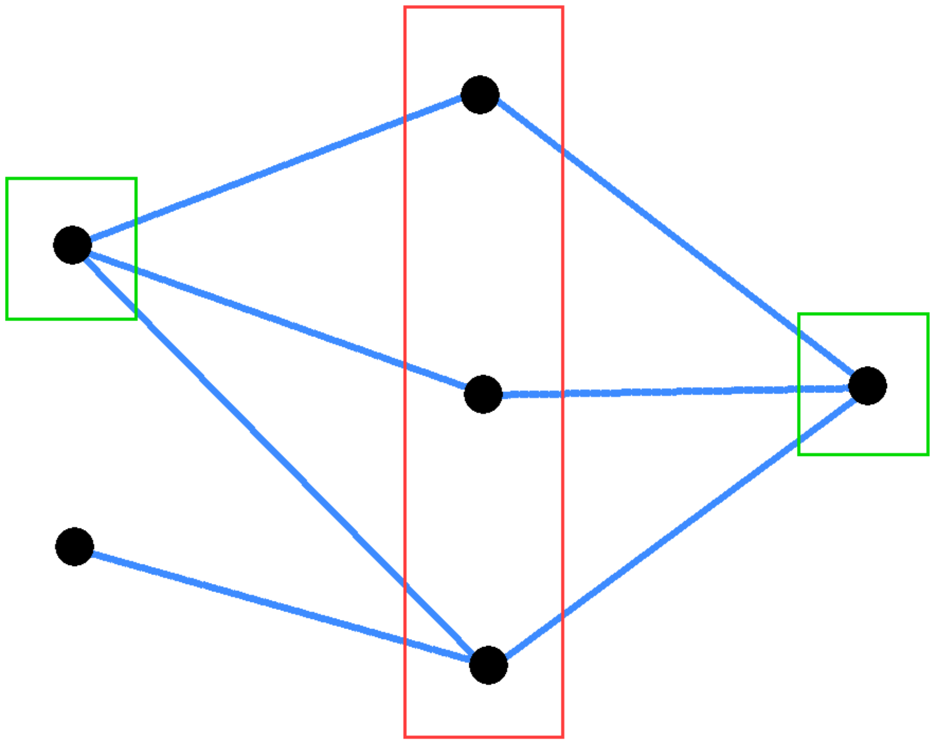
X: non empty set.
$\perp$: binary relation on X which is **irreflexive** and **symmetric**.
V: function assigning each propositional variable p to a $\perp$-**closed** subset of X.

Given $Y \subseteq X$,
$Y^{\perp} = \{x \in X | \text{for all } y \text{ in } Y,\ x \perp y\}$.
We say that Y is $\perp$-**closed** if $Y^{\perp\perp} = Y$.

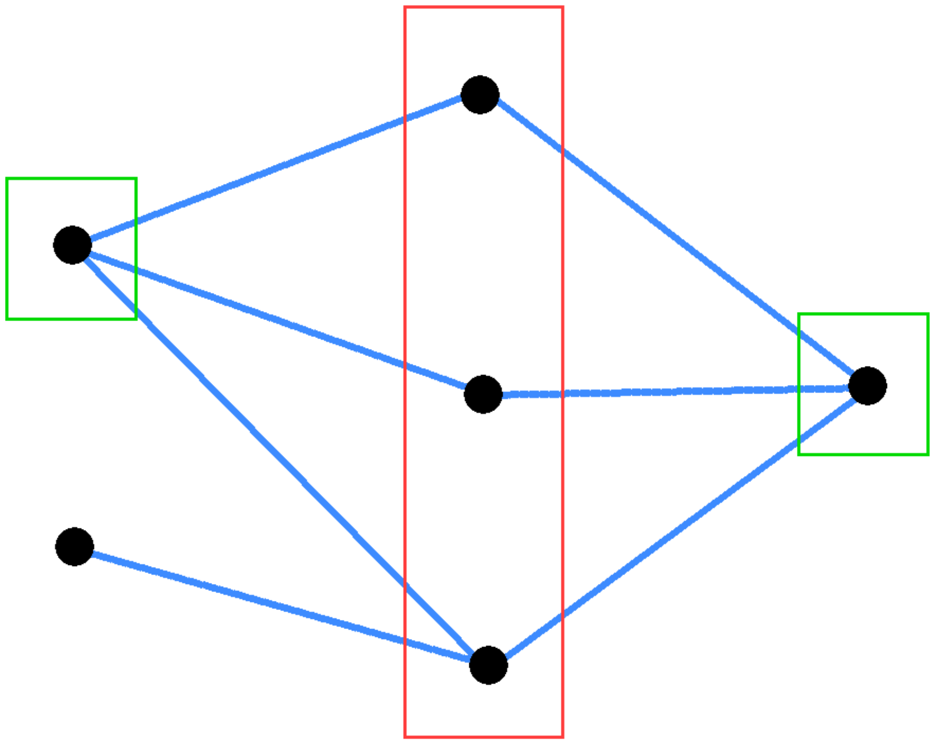




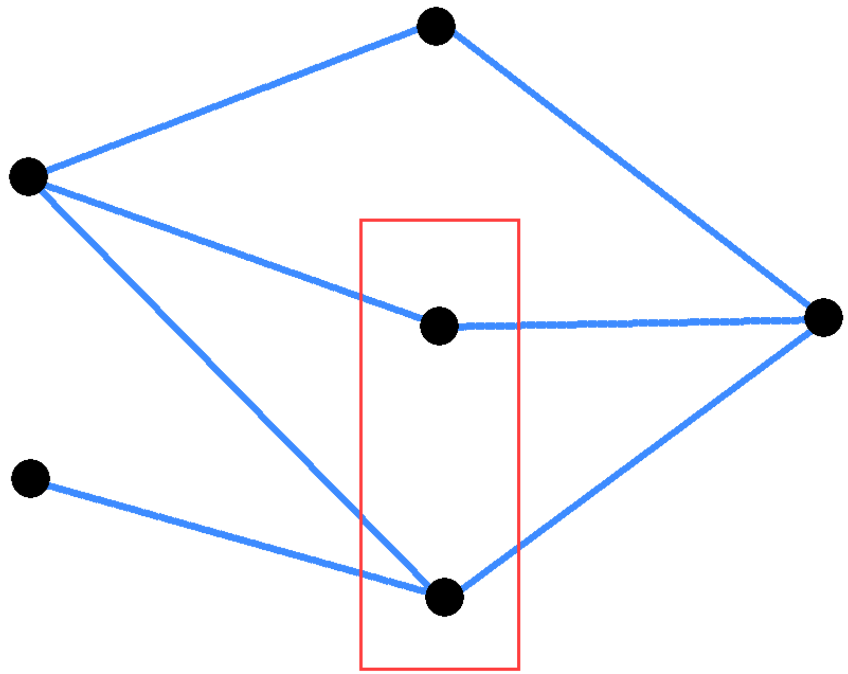
Set Y



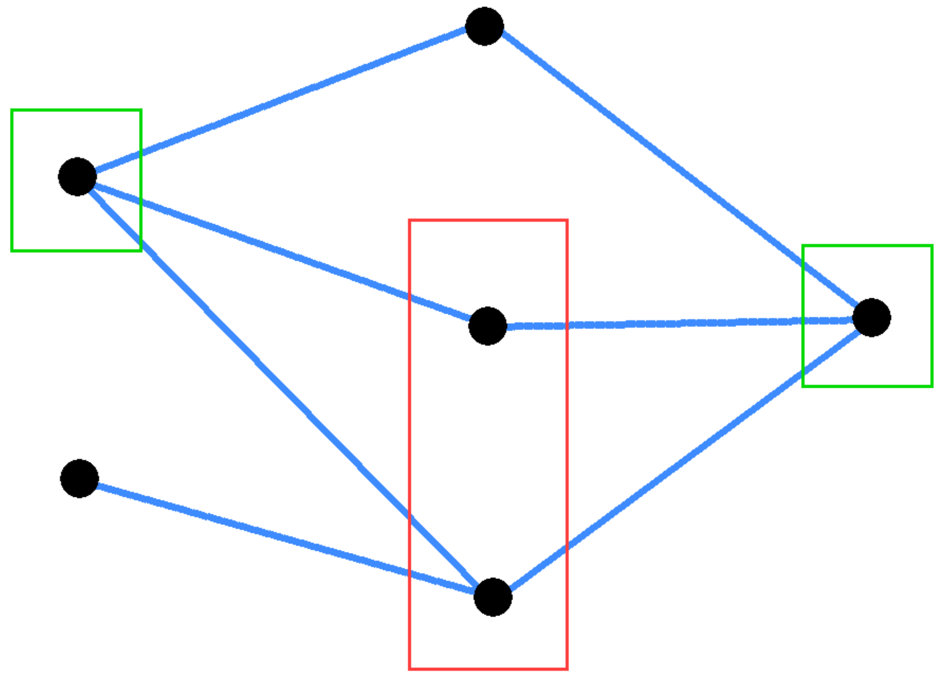
Set $\color{red}{Y}$ Set $\color{green}{Y^{\perp}}$



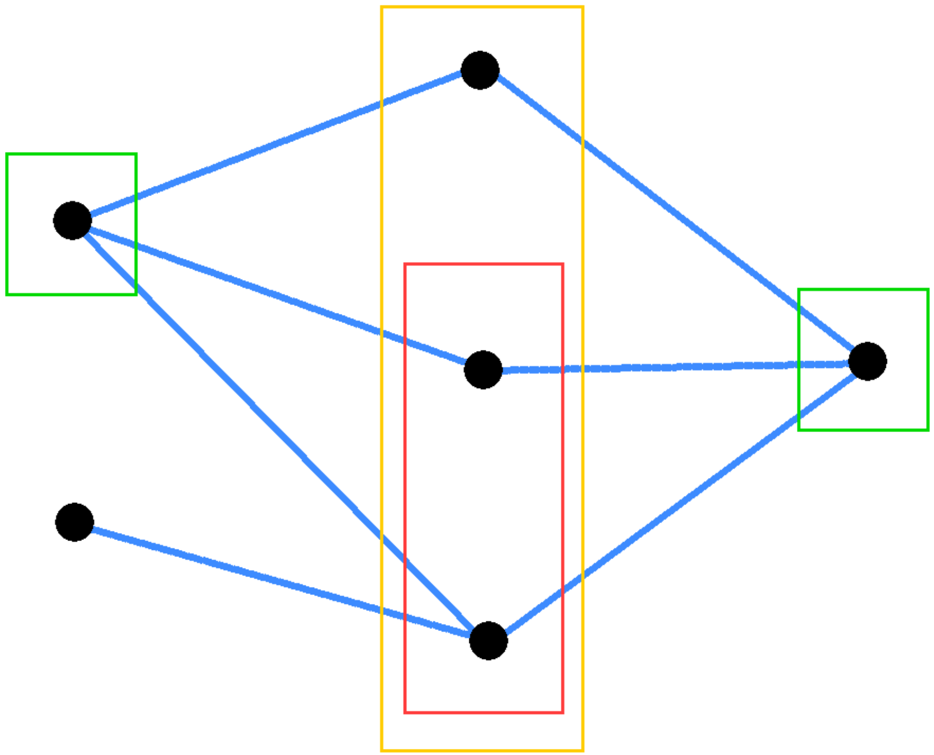
Set Y Set $Y^{\perp}$ $Y = Y^{\perp\perp}$



Set Y



Set Y Set $Y^{\perp}$



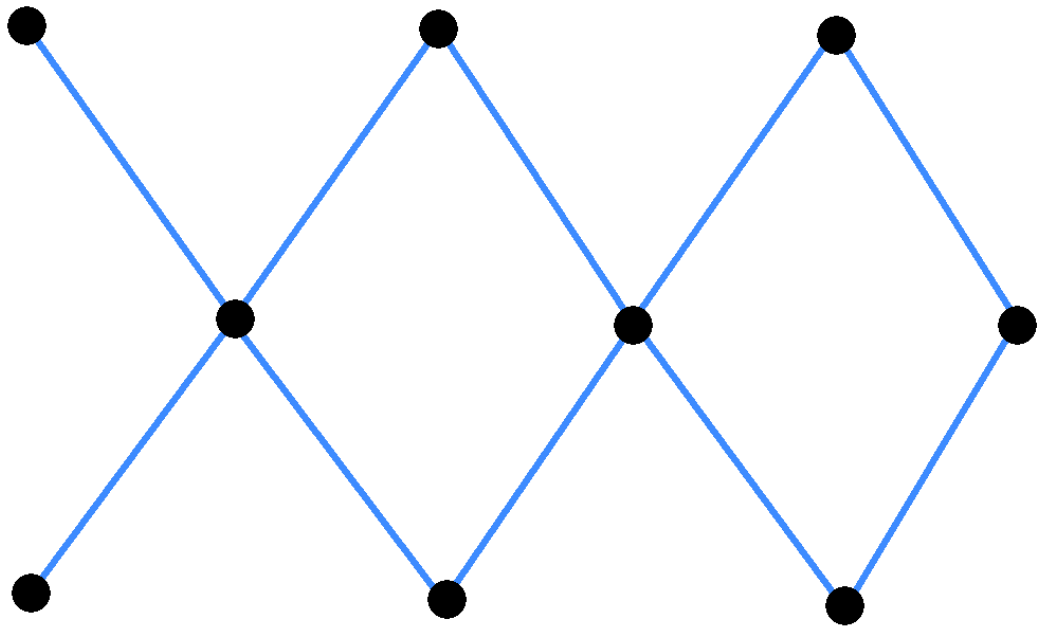
Set Y Set $Y^{\perp}$ $Y \neq Y^{\perp\perp}$

Assign formula to Kripke model

$$V(\neg A) = V(A)^{\perp}$$
$$V(A \wedge B) = V(A) \cap V(B)$$
$$V(A \to B) = \{x \in X | \forall y (x \not\perp y \text{ and } y \in V(A) \text{ then } y \in V(B))\}$$

We use $A \vee B$ as an abbreviation of $\neg(\neg A \wedge \neg B)$

Example



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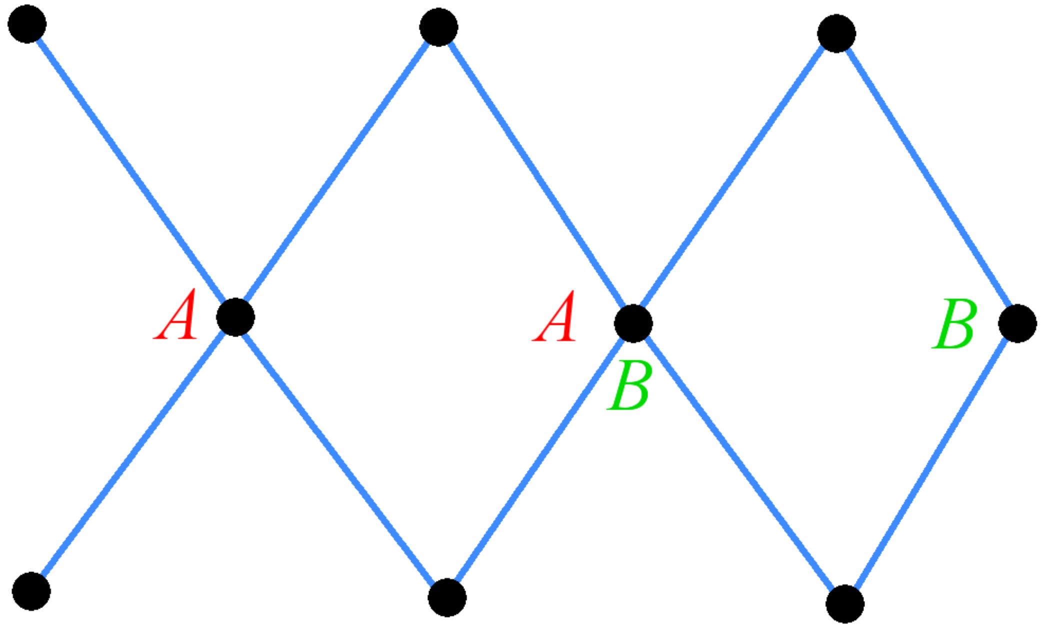

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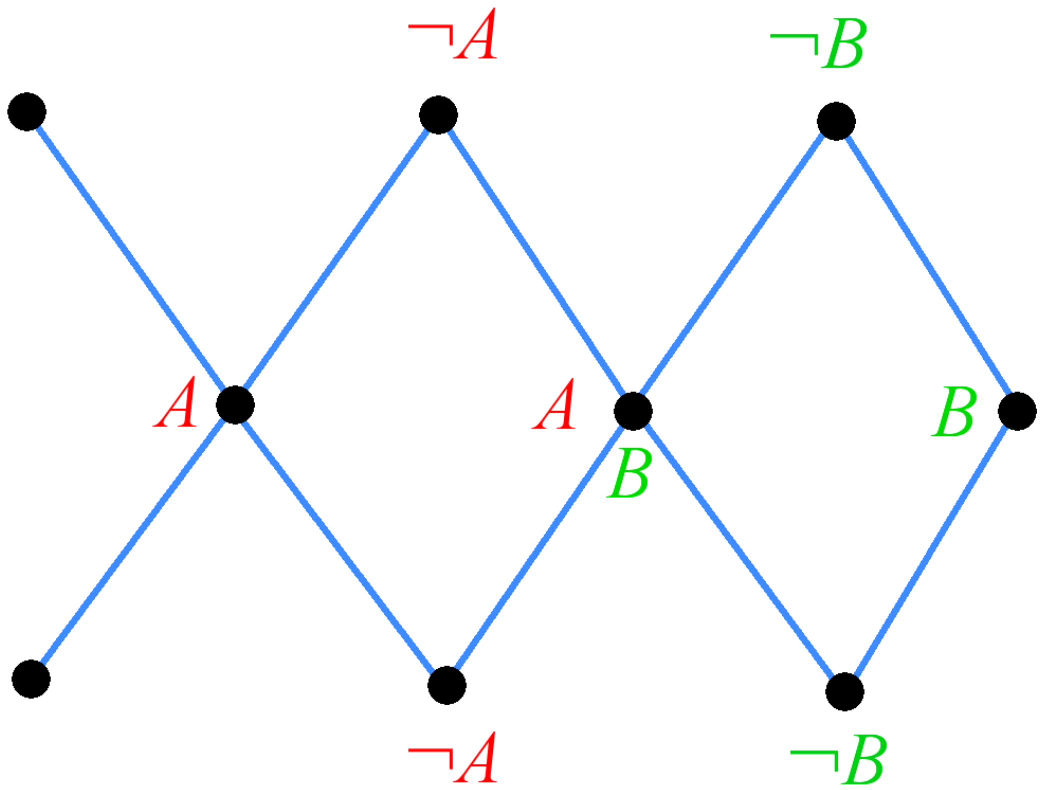

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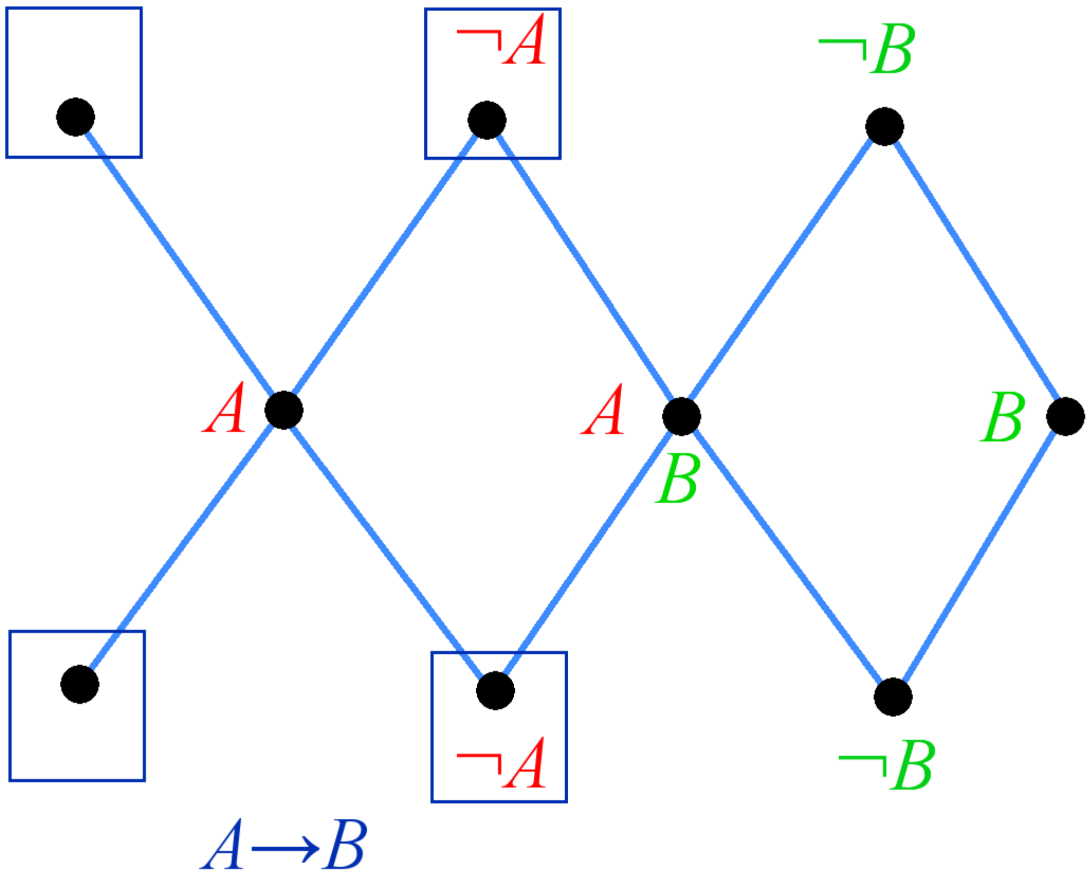
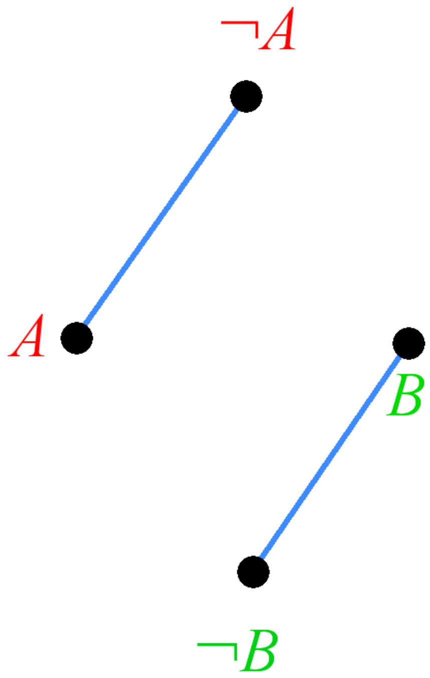




Distributive law is not satisfied in this model.

$$A \wedge (B \vee \neg A) = A \wedge \neg(\neg B \wedge A) = A$$

$$(A \wedge B) \vee (A \wedge \neg A) = \perp \vee \perp = \perp$$

2 Sequent calculi for minimal quantum logic

Sequent calculus **GO** (Nishimura 1980)

Axiom:

$$A \Rightarrow A$$

Rules:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Sigma} \text{ (extension)}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \text{ (}\wedge\text{L)} \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \text{ (}\wedge\text{L)} \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \text{ (}\wedge\text{R)}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \text{ (}\neg\text{L)} \qquad \frac{A \Rightarrow \Delta}{\neg \Delta \Rightarrow \neg A} \text{ (}\neg\text{R)}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta} \text{ (}\neg\neg\text{L)} \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A} \text{ (}\neg\neg\text{R)}$$

GO does not include implication. We can add implication rule as below. **GOI**

$$A \Rightarrow (A \to B) \to \perp, B \qquad (\to)$$

$$\frac{\Gamma_1, A \Rightarrow B, \Delta_1 \qquad \Gamma_2, A \Rightarrow B, \Delta_2 \quad \ldots \qquad \Gamma_{2^n}, A \Rightarrow B, \Delta_{2^n}}{C_1 \to D_1, C_2 \to D_2, \ldots, C_n \to D_n \Rightarrow A \to B} \ (\to\mathrm{R})$$

where, $0 \leq n$, $\Gamma_i = \{D_j | j \in \gamma(i)\}$, $\Delta_i = \{C_j | j \in \delta(i)\}$, $\langle \delta(i), \gamma(i) \rangle$ is i-th element of all divisions of $\{1, \ldots, n\}$

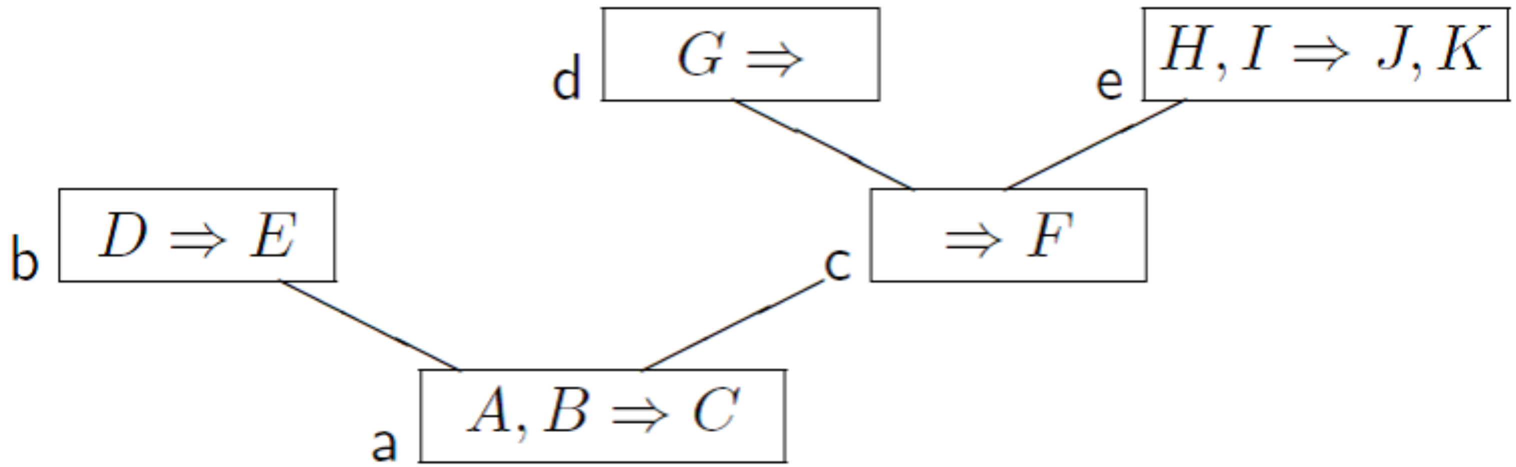
Example : if $n = 2$,

$$\frac{A \Rightarrow B, C_1, C_2 \qquad D_1, A \Rightarrow B, C_2 \qquad D_2, A \Rightarrow B, C_1 \qquad D_1, D_2, A \Rightarrow B}{C_1 \to D_1, C_2 \to D_2 \Rightarrow A \to B} \ (-$$

GO does not satisfy cut elimination.
This is the example of sequent which cannot be proved without cut.

$$
\dfrac{
  (\Rightarrow \neg\neg)\,\dfrac{
    (\Rightarrow \wedge)\,\dfrac{
      (\text{extension})\,\dfrac{p \Rightarrow p}{p, q \Rightarrow p}
      \qquad
      \dfrac{q \Rightarrow q}{p, q \Rightarrow q}\,(\text{extension})
    }{p, q \Rightarrow p \wedge q}
  }{p, q \Rightarrow \neg\neg(p \wedge q)}
  \qquad
  \dfrac{
    \dfrac{\neg(p \wedge q) \Rightarrow \neg(p \wedge q)}{r \wedge \neg(p \wedge q) \Rightarrow \neg(p \wedge q)}\,(\wedge \Rightarrow)
  }{\neg\neg(p \wedge q) \Rightarrow \neg(r \wedge \neg(p \wedge q))}\,(\Rightarrow \neg)
}{p, q \Rightarrow \neg(r \wedge \neg(p \wedge q))}\,(\text{cut})
$$

3 Labeled (Tree) sequent



Example : Labeled sequent of intuitionistic logic.
TLJ (Kashima)

axiom: $\mathsf{a} : A,\ \Gamma \Rightarrow \Delta,\ \mathsf{a} : A$ $\qquad$ $\mathsf{a} : \bot,\ \Gamma \Rightarrow \Delta$

rules:

$$\frac{\Gamma \Rightarrow \Delta,\ \mathsf{a} : A \qquad \mathsf{a} : B,\ \Gamma \Rightarrow \Delta}{\mathsf{a} : A{\to}B,\ \Gamma \Rightarrow \Delta}\ (\to 左) \qquad \frac{\mathsf{b} : A,\ \Gamma \Rightarrow \Delta,\ \mathsf{b} : B}{\Gamma \Rightarrow \Delta,\ \mathsf{a} : A{\to}B}\ (\to 右)^{(*)}$$

$$\frac{\mathsf{a} : A,\ \mathsf{a} : B,\ \Gamma \Rightarrow \Delta}{\mathsf{a} : A{\wedge}B,\ \Gamma \Rightarrow \Delta}\ (\wedge 左) \qquad \frac{\Gamma \Rightarrow \Delta,\ \mathsf{a} : A \qquad \Gamma \Rightarrow \Delta,\ \mathsf{a} : B}{\Gamma \Rightarrow \Delta,\ \mathsf{a} : A{\wedge}B}\ (\wedge 右)$$

$$\frac{\mathsf{a} : A,\ \Gamma \Rightarrow \Delta \qquad \mathsf{a} : B,\ \Gamma \Rightarrow \Delta}{\mathsf{a} : A{\vee}B,\ \Gamma \Rightarrow \Delta}\ (\vee 左) \qquad \frac{\Gamma \Rightarrow \Delta,\ \mathsf{a} : A,\ \mathsf{a} : B}{\Gamma \Rightarrow \Delta,\ \mathsf{a} : A{\vee}B}\ (\vee 右)$$

$$\frac{\mathsf{a} : A,\ \mathsf{a} : A,\ \Gamma \Rightarrow \Delta}{\mathsf{a} : A,\ \Gamma \Rightarrow \Delta}\ (\text{contraction 左}) \qquad \frac{\Gamma \Rightarrow \Delta,\ \mathsf{a} : A,\ \mathsf{a} : A}{\Gamma \Rightarrow \Delta,\ \mathsf{a} : A}\ (\text{contraction 右})$$

$$\frac{\mathsf{b} : A,\ \Gamma \Rightarrow \Delta}{\mathsf{a} : A,\ \Gamma \Rightarrow \Delta}\ (遺伝性)^{(**)}$$

4 Labeled sequent of quantum logic

TGOI

・Axiom and rules for $\wedge$ is same to **TLJ**. There is no rule for $\vee$ as $\vee$ is an abbreviation.

$$\frac{\Gamma \Rightarrow \Delta, b : A \quad b : B, \Gamma \Rightarrow \Delta}{a : A \rightarrow B, \Gamma \Rightarrow \Delta}\ (\rightarrow \mathsf{L})$$

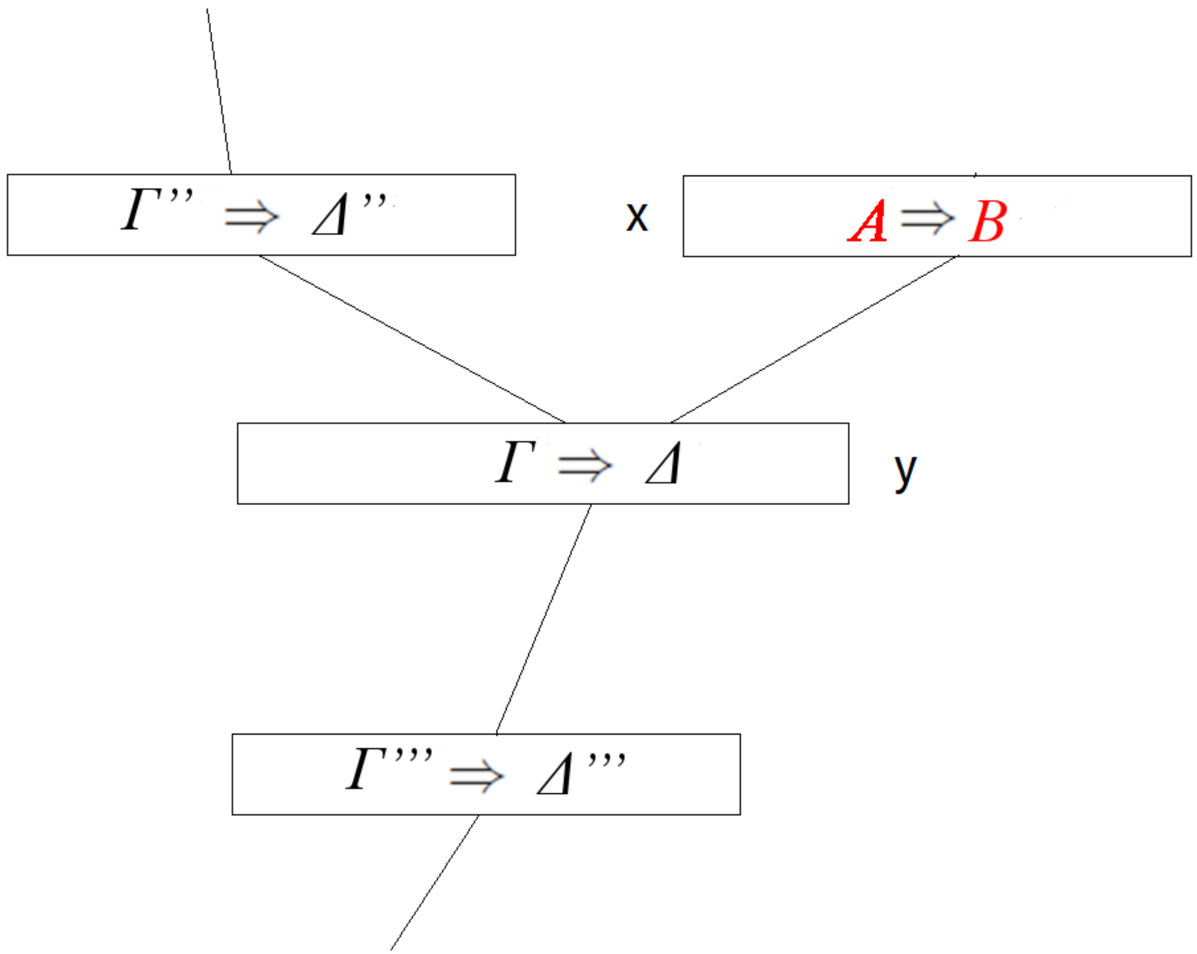
$$\frac{b : A, \Gamma \Rightarrow \Delta, b : B}{\Gamma \Rightarrow \Delta, a : A \rightarrow B}\ (\rightarrow \mathsf{R}) \qquad \frac{b : \neg A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, a : A}\ (\neg)$$

$(\rightarrow L)$: b and a are related by $\not\perp$.

$(\rightarrow R), (\neg)$: b and a are related by $\not\perp$ and only A and B or $\neg A$ exists in b. b is deleted in lower sequent.

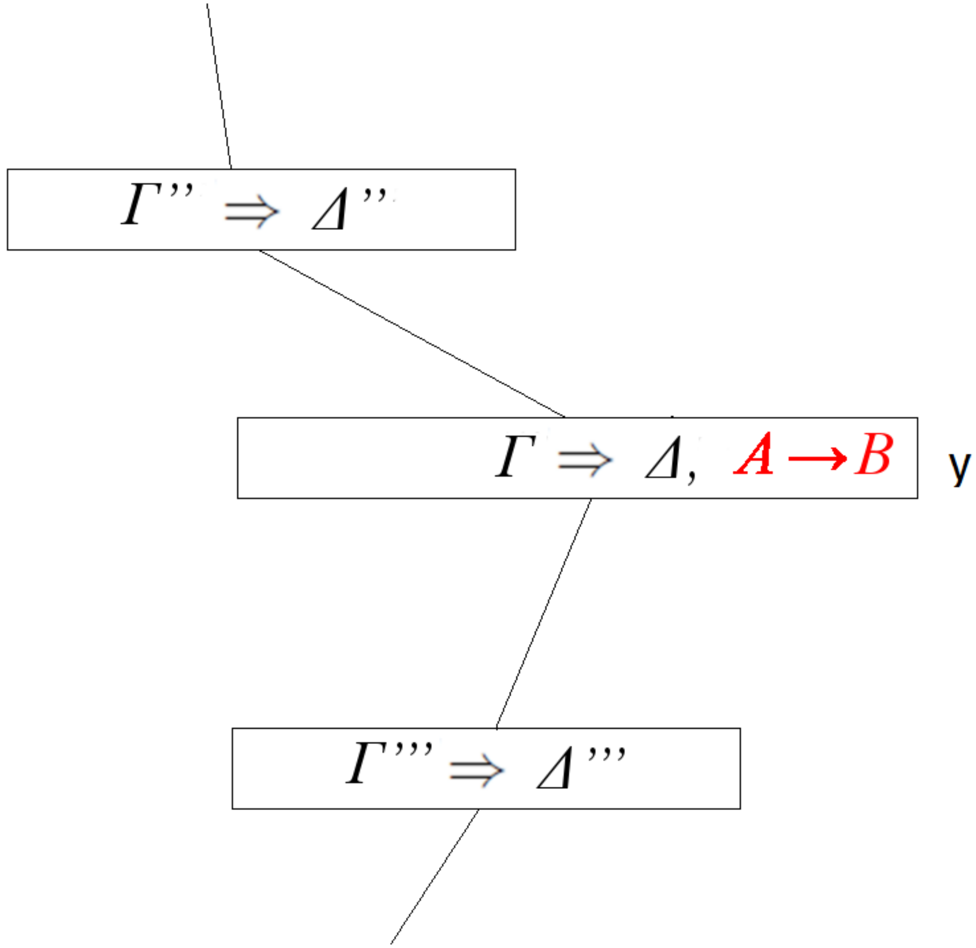
$$\frac{\Gamma \Rightarrow \Delta, a : A \quad a : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ (cut)}$$

$(\to R)$ before



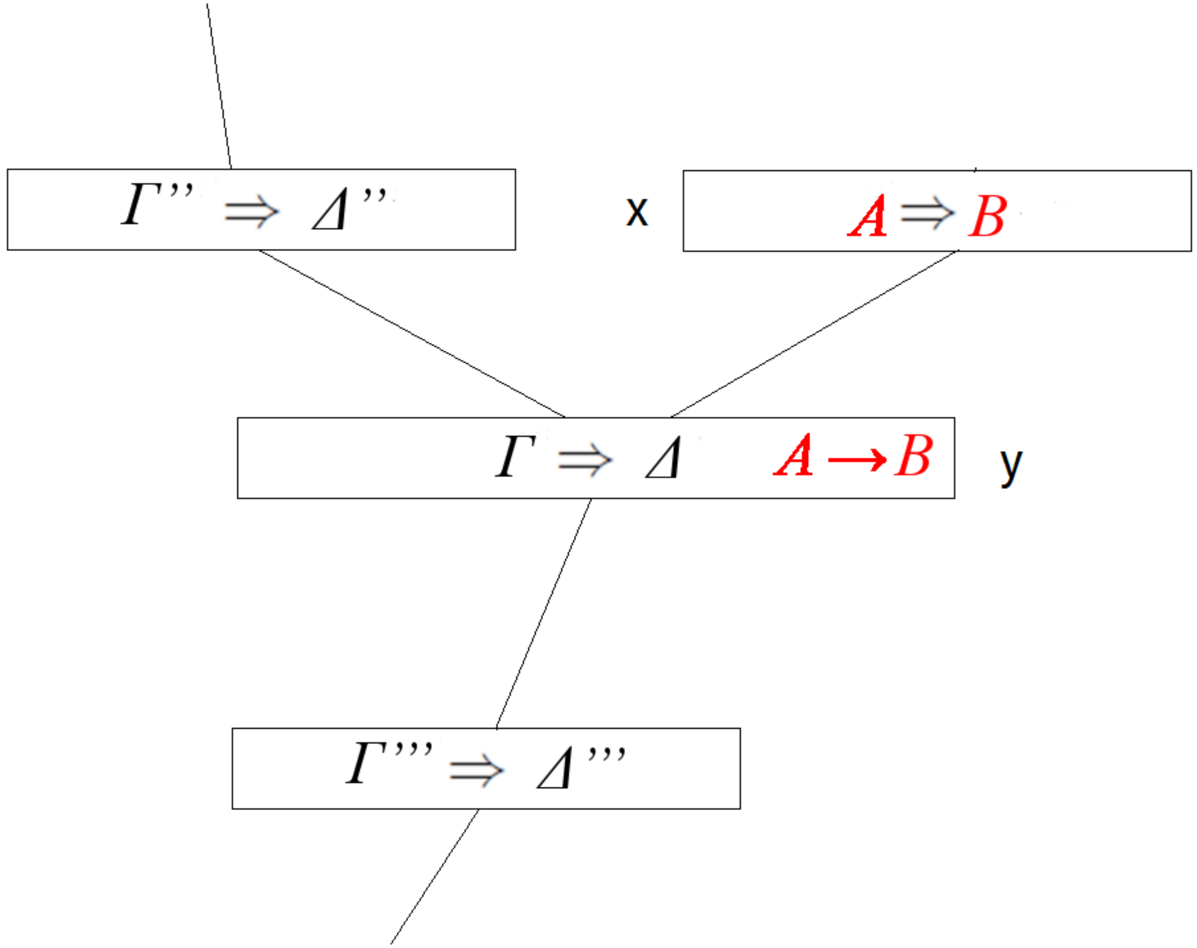
(In these pictures, relation is $\not\perp$. Not $\perp$.)

$(\to R)$ after



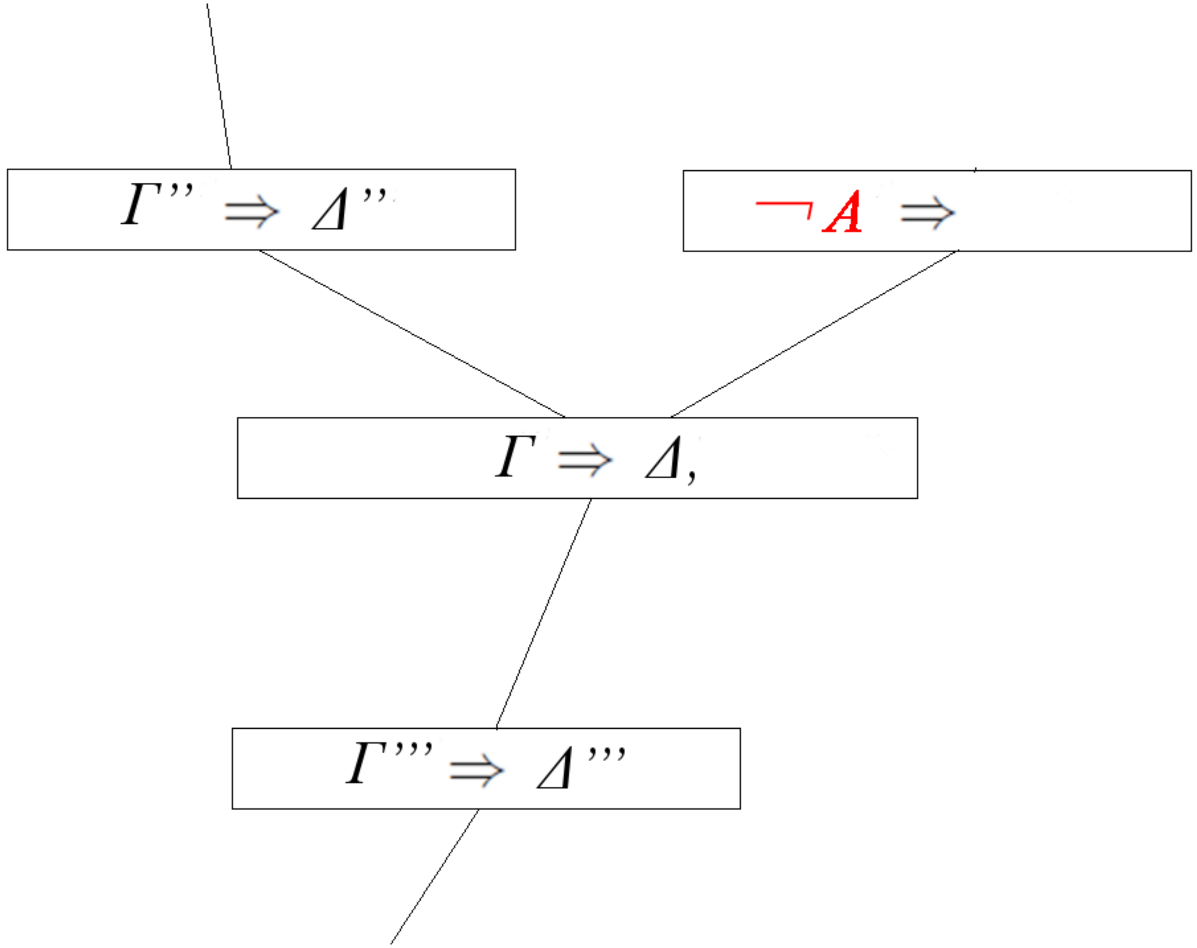
(In these pictures, relation is $\not\perp$. Not $\perp$.)

$(\rightarrow R)$ before 2



(In these pictures, relation is $\not\perp$. Not $\perp$.)

$(\neg)$ before



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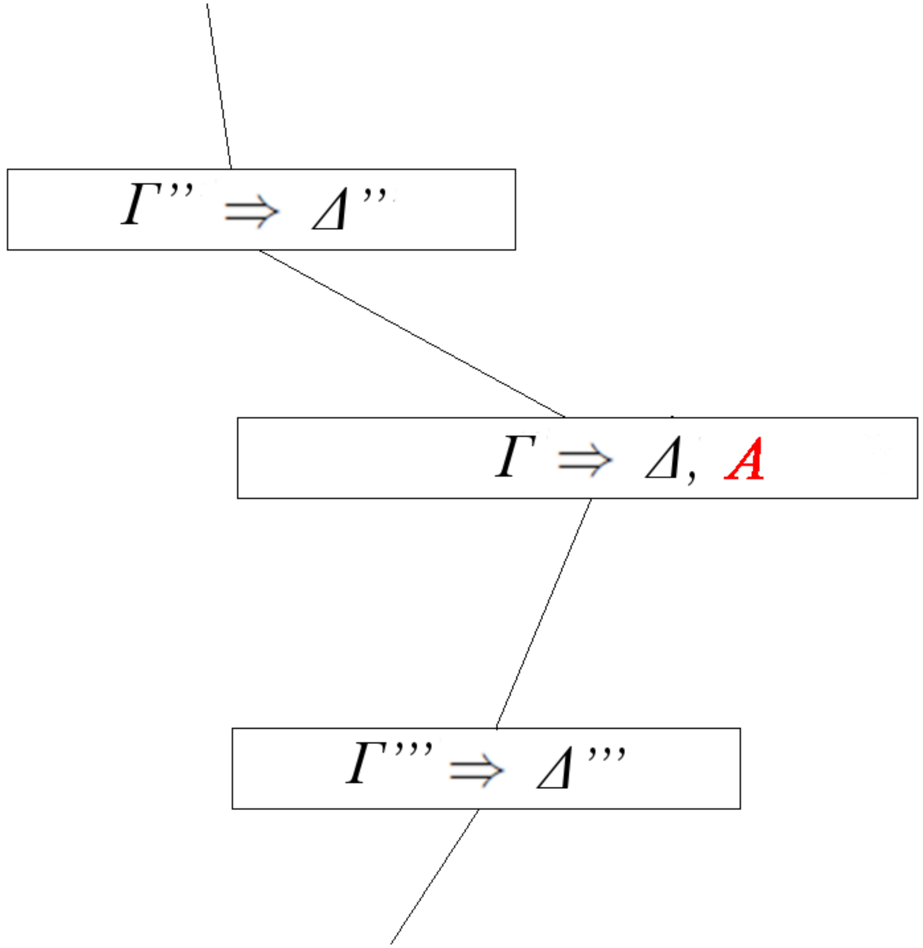
$(\neg)$ after



(In these pictures, relation is $\not\perp$. Not $\perp$.)

Completeness of **TGOI**

Theorem

> If **TGOI** $\nvdash \Rightarrow a : A$ (only one node a exists),
> then, there exist O-model $(X, \perp, V)$ and $x \in X$
> which satisfy $x \not\models A$.

We can see that this sequent system has cut elimination theorem because we can prove completeness without cut rule.

Proof:
We made $(X, \perp, V)$ by the 2 steps algorithm.

step 1

・ Expand the frame with preserving the unprovability without the $(\neg)$ rule. We continue this step until no rules can be apply.

step 2

・Expand the frame with $(\neg)$ rure. We only apply this to all **propositional variable** in right side of sequent.

We continue these 2 steps until it become to **practical end state**. $1 \to 2 \to 1 \to 2 \to 1 \to \ldots$

(In this algorithm, there is infinite loop. So we have to terminate this somewhere. Fortunately, there is a state that meaning of which is no more changeable. We call this **practical end state**.)

$$\boxed{\Rightarrow \neg(\neg(A \to B) \wedge \neg(B \to A))}$$

$$\Rightarrow \neg(\neg(A \rightarrow B) \wedge \neg(B \rightarrow A))$$

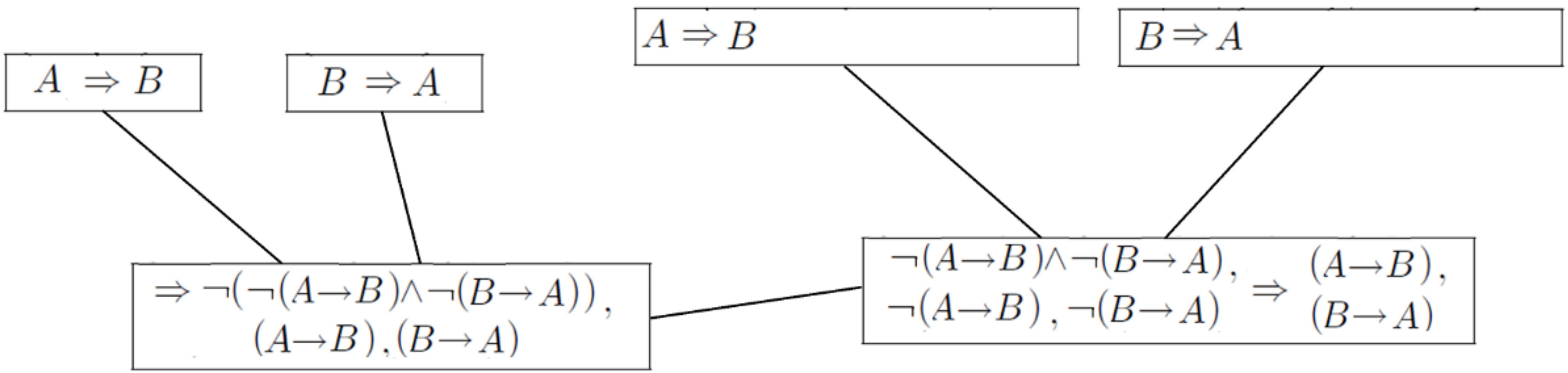
$$\neg(A \rightarrow B) \wedge \neg(B \rightarrow A) \Rightarrow$$

$$\boxed{\Rightarrow \neg(\neg(A \to B) \wedge \neg(B \to A))}$$

$$\boxed{\begin{array}{l} \neg(A \to B) \wedge \neg(B \to A), \\ \neg(A \to B), \neg(B \to A) \end{array} \Rightarrow}$$

$$\boxed{\Rightarrow \neg(\neg(A\to B)\wedge\neg(B\to A)),\ (A\to B),(B\to A)}$$

$$\boxed{\neg(A\to B)\wedge\neg(B\to A),\ \neg(A\to B),\neg(B\to A) \Rightarrow (A\to B),\ (B\to A)}$$



$A \Rightarrow B$
$B \Rightarrow A$
$\Rightarrow \neg(\neg(A\to B)\wedge\neg(B\to A)),\ (A\to B),(B\to A)$
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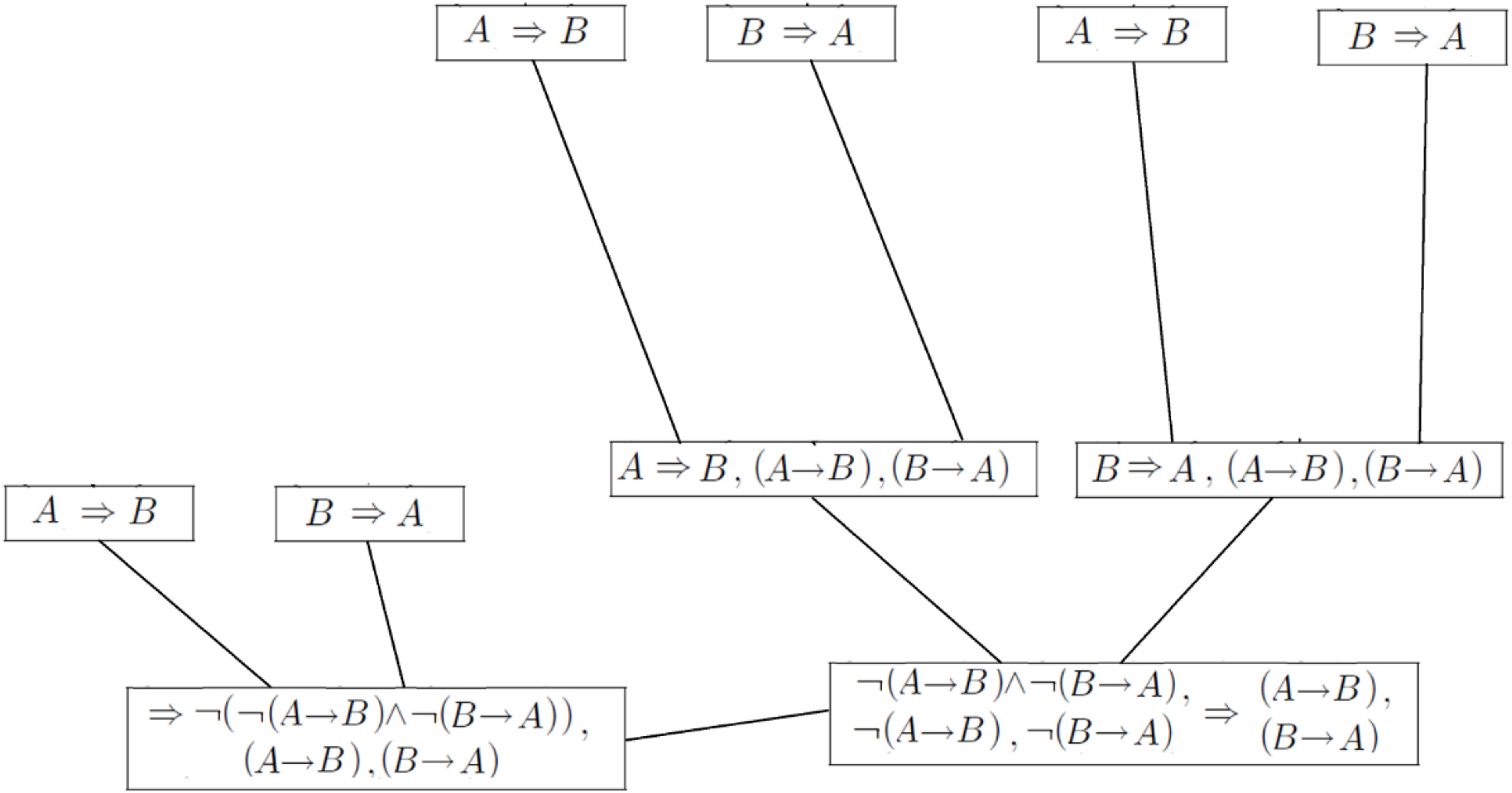

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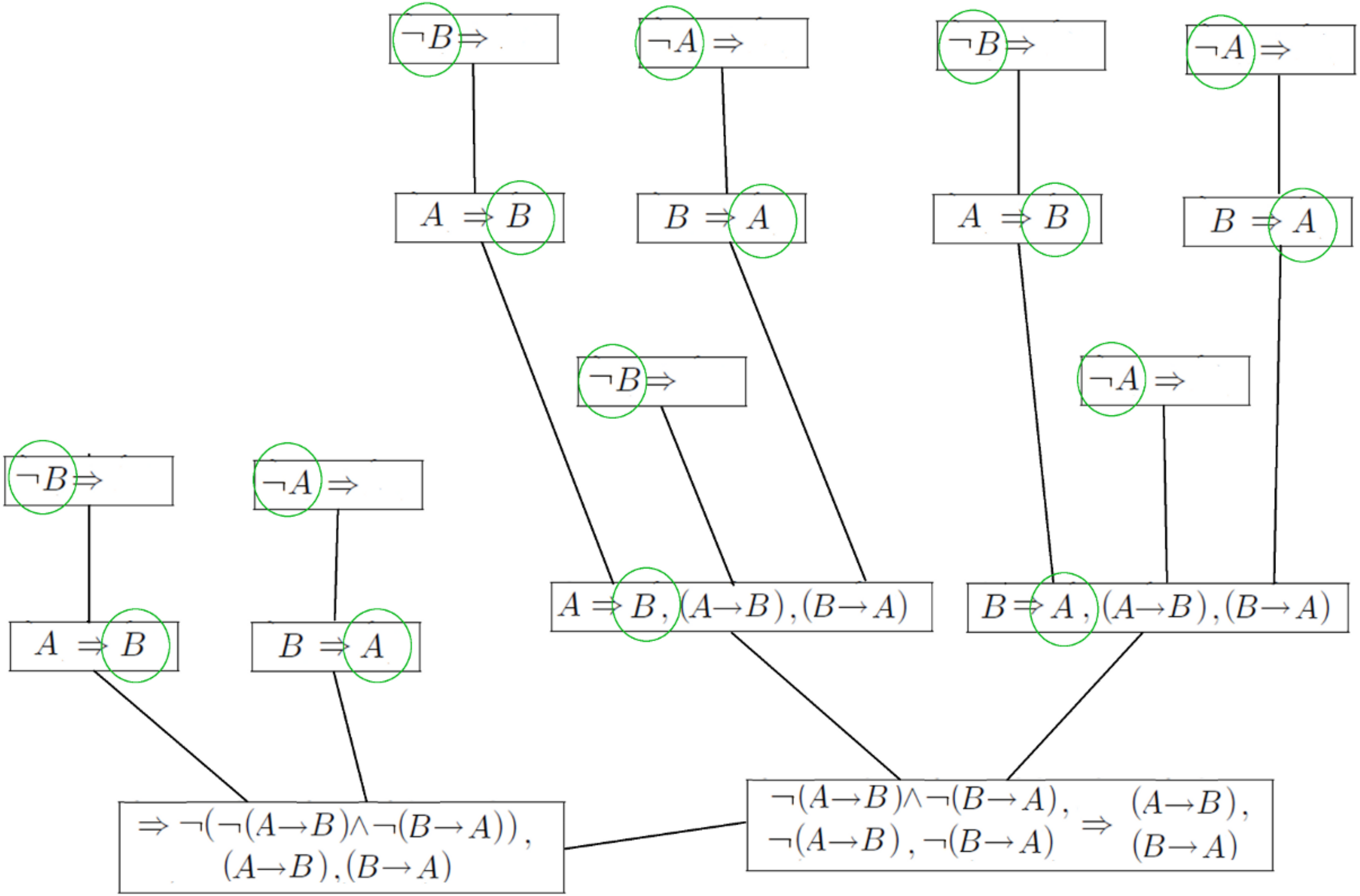

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A ⇒ B, (A→B), (B→A)
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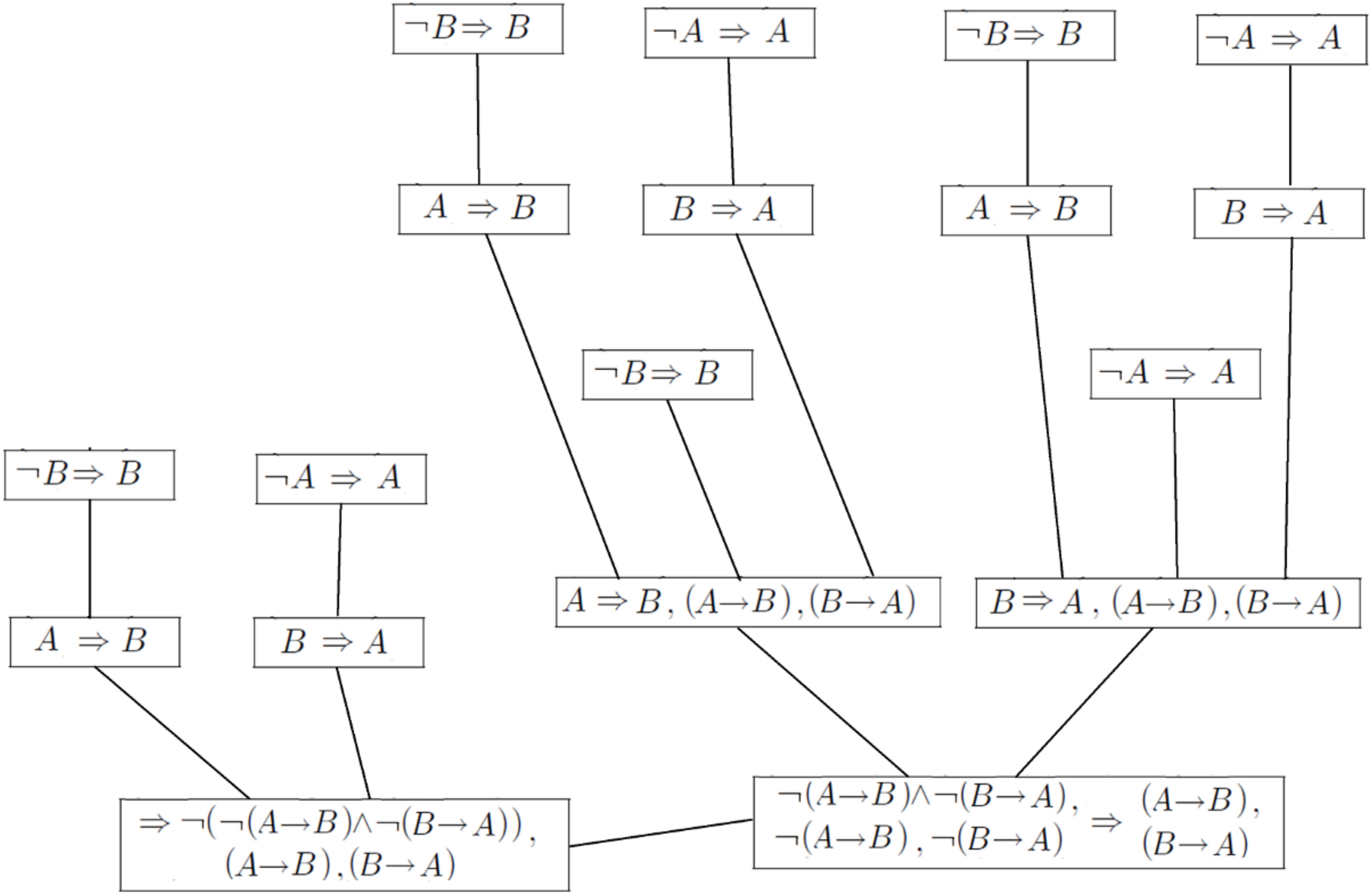
⇒ ¬(¬(A→B)∧¬(B→A)), (A→B), (B→A)
¬(A→B)∧¬(B→A), ¬(A→B), ¬(B→A) ⇒ (A→B), (B→A)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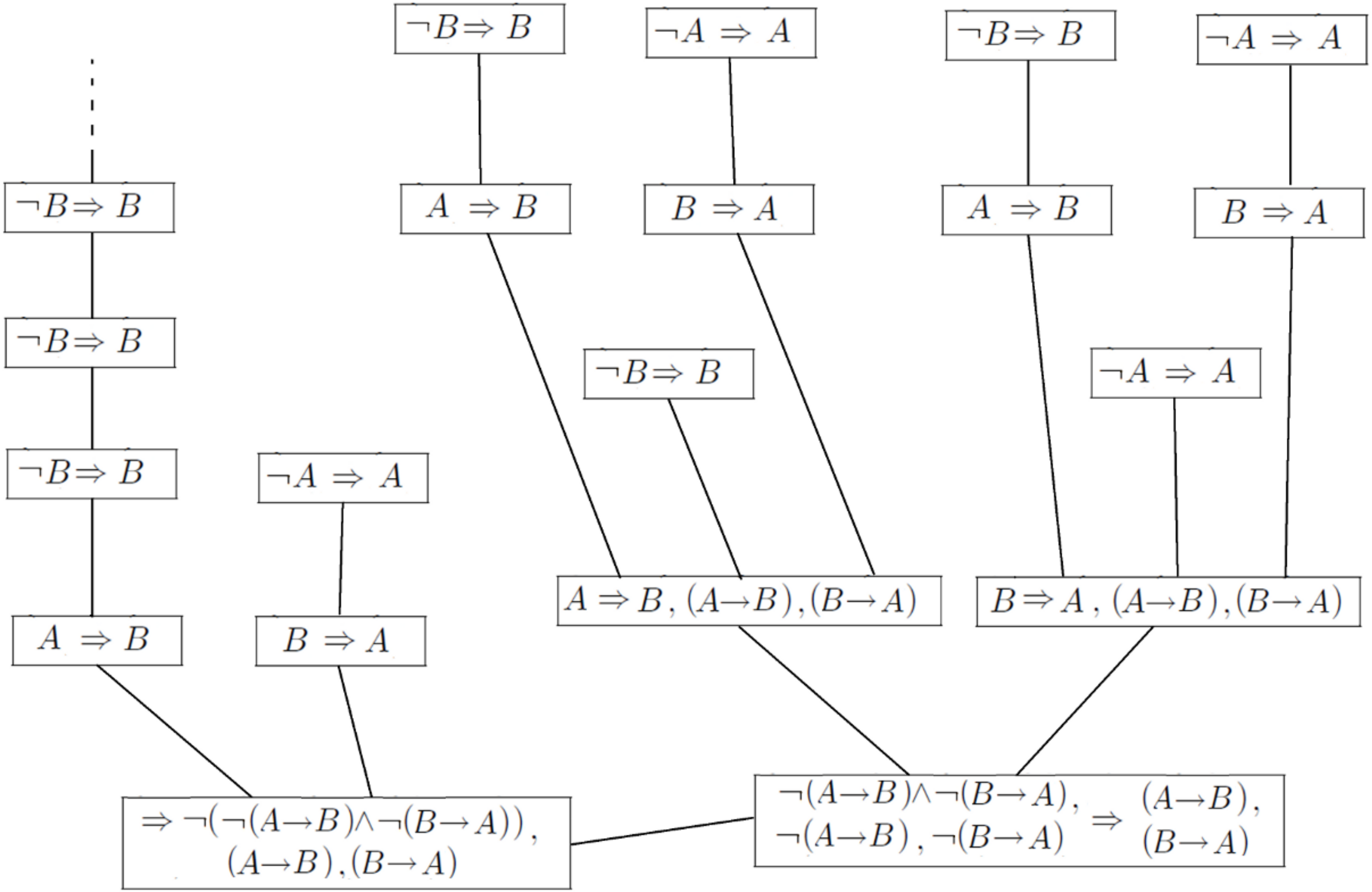
$\Rightarrow \neg(\neg(A \to B) \wedge \neg(B \to A)), (A \to B), (B \to A)$

If we continue this, we are going to infinite loop like next page. I omit the detail but the meaning of this model is same to the model in previous page. So, the model in previous page is practical end state.

The red circles are the point which we want to attach $V(A)$. Although the set of red $\bigcirc$ is not $\perp$-closed set , it can be extended to the $\perp$-closed set by lemma. Blue points are $V(\neg A)$. Furthermore, there is no contradict point like $A \Rightarrow A$ or $\neg A \Rightarrow \neg A$.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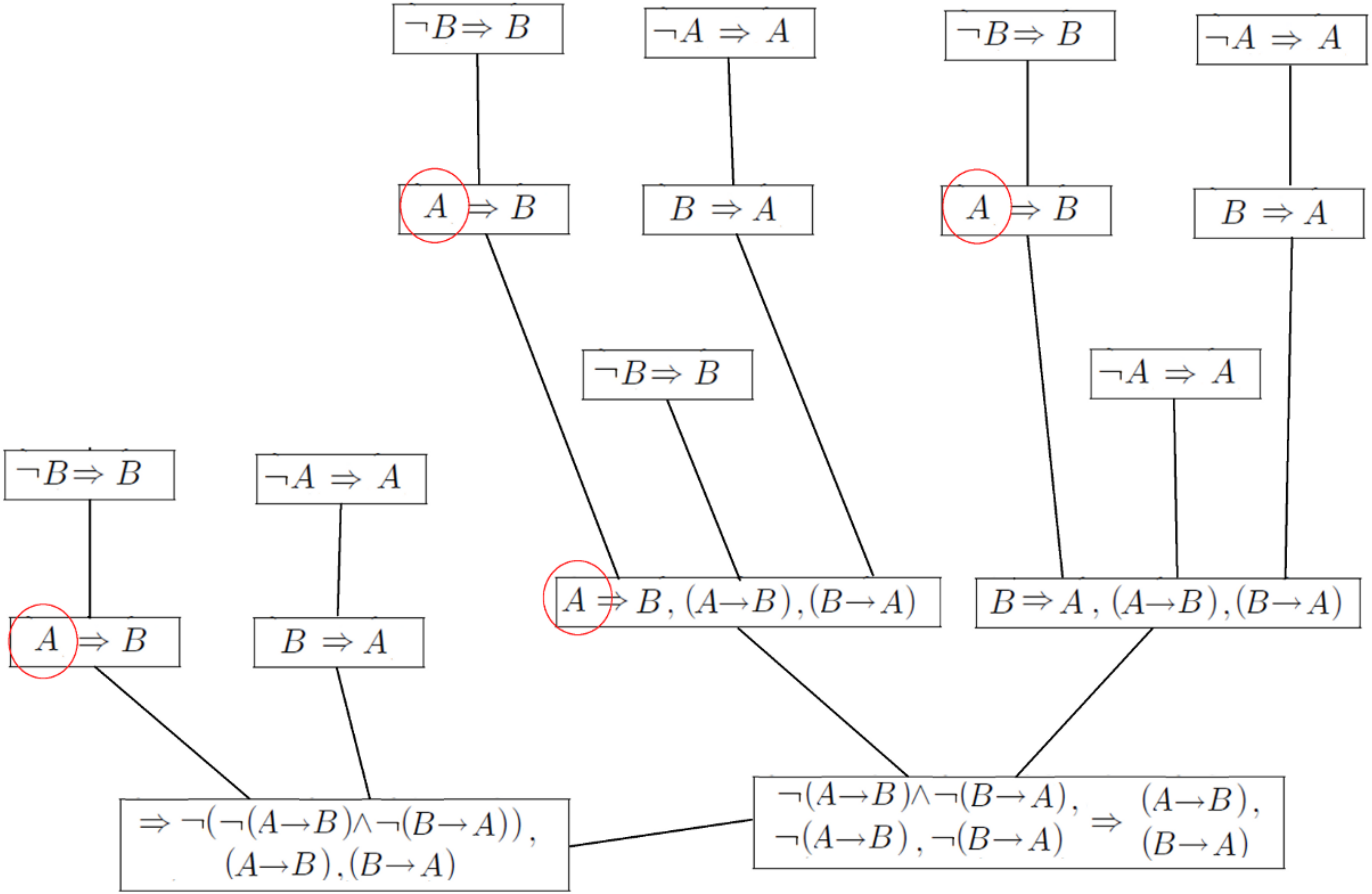

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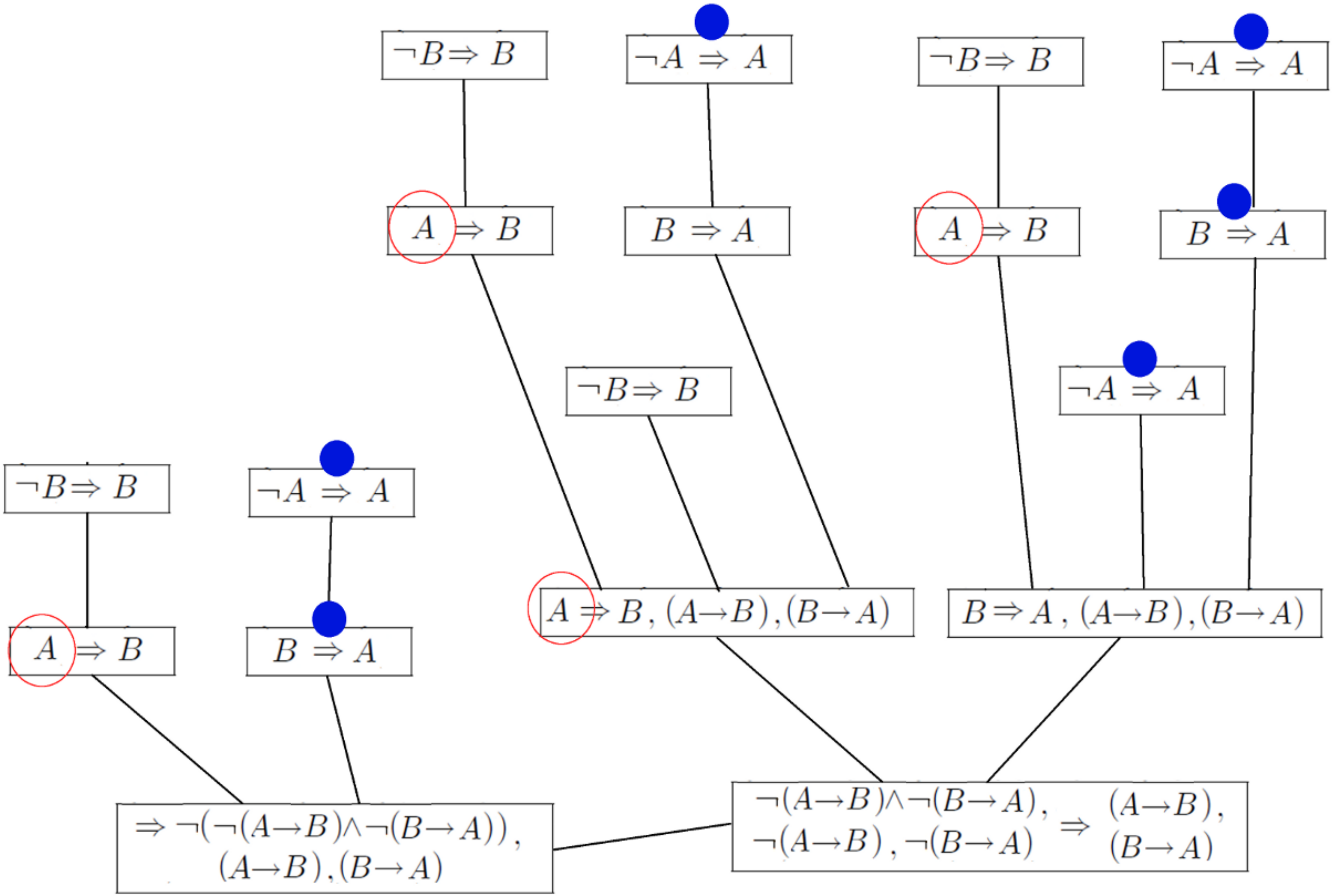

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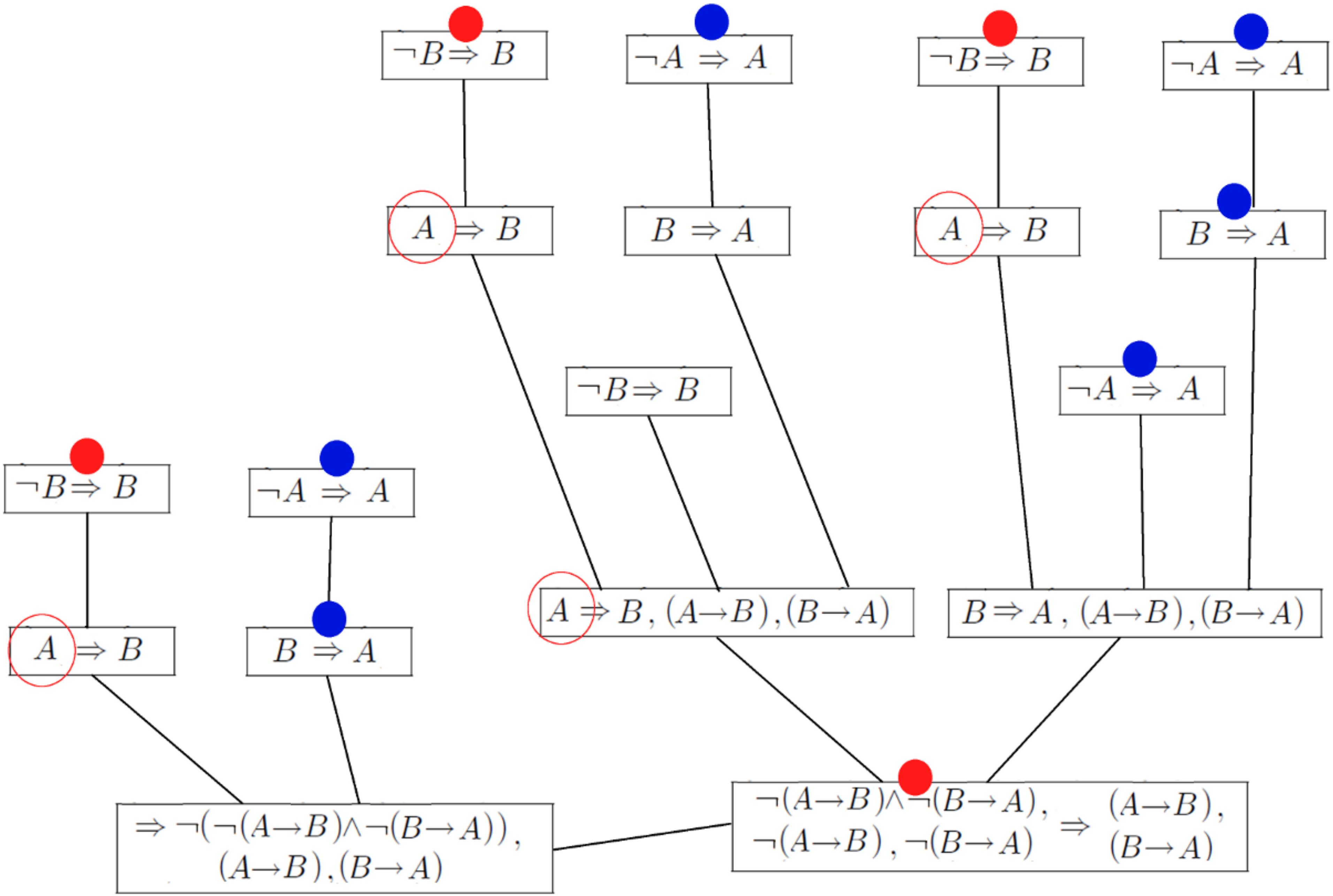

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$A \Rightarrow B$
$B \Rightarrow A$
$\neg(A \rightarrow B) \wedge \neg(B \rightarrow A), \neg(A \rightarrow B), \neg(B \rightarrow A) \Rightarrow (A \rightarrow B), (B \rightarrow A)$
$\Rightarrow \neg(\neg(A \rightarrow B) \wedge \neg(B \rightarrow A)), (A \rightarrow B), (B \rightarrow A)$

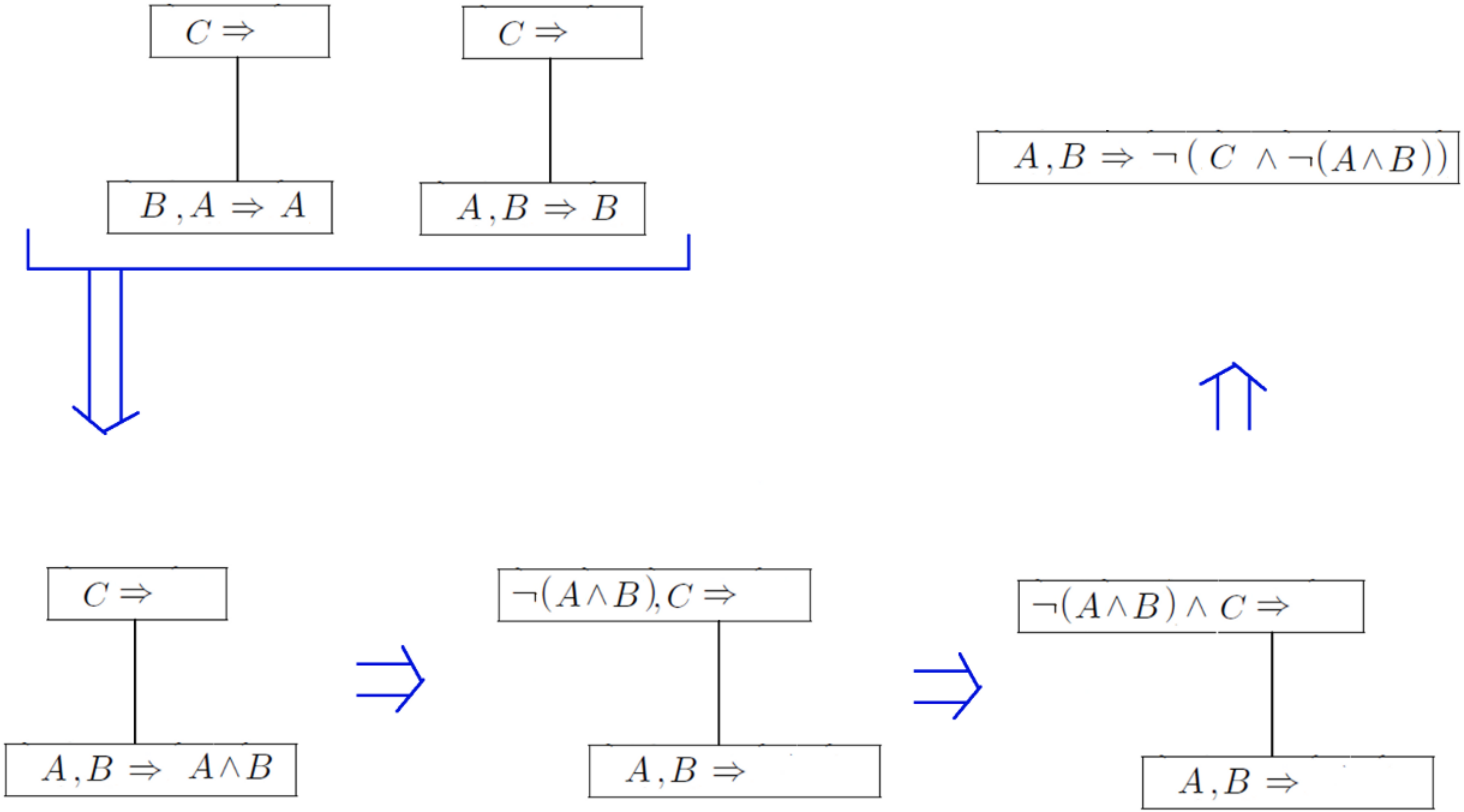
Theorem

TGOI $\vdash \Rightarrow a : A$ (only one node a exists).

$\Leftrightarrow$

GOI $\vdash \Rightarrow A$

We can prove the formula which we saw before without cut rule.



$C \Rightarrow$
$B, A \Rightarrow A$
$C \Rightarrow$
$A, B \Rightarrow B$
$A, B \Rightarrow \neg(C \wedge \neg(A \wedge B))$
$C \Rightarrow$
$A, B \Rightarrow A \wedge B$
$\neg(A \wedge B), C \Rightarrow$
$A, B \Rightarrow$
$\neg(A \wedge B) \wedge C \Rightarrow$
$A, B \Rightarrow$

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Thank you for listening !