Reverse Mathematics, Artin-Wedderburn Theorem, and Rees Theorem

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Contents

1 Reverse Mathematics

Subsystems of second order arithmetic Z_2

system	characteristic axiom
$\mathsf{RCA_0}^*$	recursive comprehension axiom and Σ^0_0-induction
$\mathsf{RCA_0}$	recursive comprehension axiom and Σ^0_1-induction
$\mathsf{WKL_0}$	weak König's lemma
$\mathsf{ACA_0}$	arithmetical comprehension axiom
$\mathsf{ATR_0}$	arithmetical transfinite recursion
Π^1_1-$\mathsf{CA_0}$	Π^1_1-comprehension axiom

The main stream of Reverse Mathematics aims at

- formalizing mathematical theorems in the weak subsystem $\mathsf{RCA_0}$ of second order arithmetic Z_2,
- and classifying mathematical theorems into several subsystems of Z_2 in terms of set existence axioms exactly needed to prove them (cf. Simpson, [7]).

RM and structural theorems for groups

Theorem 1.1.

1. Over RCA_0^*, RCA_0 is equivalent to the fundamental theorem of finitely generated countable abelian groups (18c) (Hatzikiriakou (1989), [5]).
2. Over RCA_0, ACA_0 is equivalent to the statement that every countable abelian group is the direct sum of a torsion group and a torsion-free group (Friedman, Simpson, and Smith (1983), [4]).
3. Over RCA_0, ATR_0 is equivalent to the Ulm's theorem (1933) for countable abelian groups (-).
4. Over RCA_0, Π^1_1-CA_0 is equivalent to the statement that every countable abelian group is the direct sum of a divisible group and a reduced group (-).

2 Artin-Wedderburn theorem for rings

Definition 1. A ring R is said to be *simple* if

$$(\forall a \in R)(\forall b \in R \setminus \{0_R\})(\exists x, y \in R)(a = xby).$$

If a ring R is simple then R does not have any non-trivial proper ideal.

Definition 2. A ring R is said to be *semisimple* if R is isomorphic to the finite product of simple rings.

Definition 3. A ring R is said to be *left Artinian* if there does not exists an infinite strictly descending chain of left ideals

$$I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_n \supsetneq \cdots .$$[1]

Definition 4. The *Jacobson radical* $\mathrm{Jac}(R)$ of a ring R is defined as

$$\mathrm{Jac}(R) = \{r \in R : (\forall a \in R)(\exists b \in R)[(1_R - ra)b = 1_R]\}.$$

[1] It is interesting to consider the more strong chain condition that there does not exists an infinite sequence of elements $\langle a_i : i \in \mathbb{N}\rangle$ such that $(\forall i)(a_{i+1} \in (a_i) \wedge a_i \notin (a_{i+1}))$.

Theorem 2.1 (Wedderburn (1907)-Artin (1927))**.** Let R be a ring. The following are equivalent.

1. R is left Artinian and $\mathrm{Jac}(R) = \{0_R\}$.
2. R is semisimple, i.e., there exists simple rings $R_0, R_1, \ldots, R_n$ such that

$$R \cong R_0 \oplus R_1 \oplus \cdots \oplus R_n.$$

3. R is isomorphic to the finite product of matrix rings over division rings, i.e., there exists division rings $D_0, D_1, \ldots, D_n$ and positive integers $m_0, m_1, \ldots, m_n$ such that

$$R \cong M_{m_0}(D_0) \oplus M_{m_1}(D_1) \oplus \cdots \oplus M_{m_n}(D_n).$$

- Wedderburn's part is $2 \leftrightarrow 3$.
- Artin's part is $1 \leftrightarrow 2$.

3 Artin-Wedderburn theorem and WKL_0

Proposition 3.1. Wedderburn's part of the theorem for countable rings is provable in RCA_0.

Theorem 3.2 (Conidis (2012), [1,2])**.** Every Artinian commutative ring is isomorphic to a finite direct product of local Artinian commutative rings.

The result above is based on the result below.

Theorem 3.3 (Downey, Lemmp, and Mileti (2007), [3])**.** Over RCA_0, WKL_0 is equivalent to the statement that every commutative ring which is not a field has a non-trivial proper ideal.

Corollary 3.4. Artin's part of the theorem for countable rings implies WKL_0 over RCA_0.

It is likely that WKL_0 proves Artin's part.

Summary; RM for Artin-Wedderburn theorem and Rees theorem

theorem	date	classified into
Wedderburn's theorem	1907	RCA_0
Artin's generalization	1927	$\approx \mathsf{WKL}_0$
Rees theorem	1940	$\approx \mathsf{ACA}_0$

- "The algebraists began to analyze Wedderburn's theorem and tried to find an even more abstract back ground." (Artin)
- "The Rees Theorem, strongly motivated by Wedderburn-Artin Theorem for rings..." (Howie, [6])

More abstract theories we explore, stronger axioms are needed to make statements nonvacuous.

4 Rees theorem for semigroups

For convenience, we assume that a semigroup does not contain the 0-element.

Definition 5. A semigroup S is said to be *simple* if $(\forall a, b \in S)(\exists x, y \in S)(a = xby)$. If a semigroup S is simple then S does not have any non-trivial proper ideal.

Definition 6. We define an order on the set of idempotents of a semigroup as $f \leq e \Leftrightarrow ef = fe = f$. A semigroup is said to be *complete* if there exists a minimal idempotent with respect to the order.[2]

Definition 7. Let I, Λ be non-empty sets, G be a group, and $P : \Lambda \times I \to G$. The *Rees matrix semigroup* $M(G; I, \Lambda, P)$ is the set $I \times G \times \Lambda$ together with the multiplication $(i, g, \lambda) \cdot (j, h, \mu) = (i, gP_{\lambda j}h, \mu)$.

Theorem 4.1 (Rees (1940))**.** If a semigroup S is simple and complete then there exist non-empty sets I, Λ, a group G, and $P : \Lambda \times I \to G$ such that $S \cong M(G; I, \Lambda, P)$, and vice varsa.

[2]This can be seen as a kind of chain condition.

5 Formalizing the proof of Rees theorem in ACA_0

Definition 8. The following is defined in RCA_0. Let S be a countable semigroup. A binary relation $\mathcal{L}$ on S is said to be the *left equivalence* if

$$\mathcal{L} = \{(a, b) \in S \times S : (\exists x, y \in S)(a = xb \wedge b = ya)\}.$$

The *right equivalence* $\mathcal{R}$ is defined similarly. Note that the condition of the right-hand-side is Σ^0_1.

Lemma 5.1. The following are equivalent over RCA_0.

1. ACA_0.
2. Let $\varphi(x, y) \in \Sigma^0_1$ be an equivalence relation on a set $A \subset \mathbb{N}$, i.e.,
   - $(\forall a \in A)(\varphi(a, a))$,
   - $(\forall a, b \in A)(\varphi(a, b) \to \varphi(b, a))$,
   - $(\forall a, b, c \in A)(\varphi(a, b) \land \varphi(b, c) \to \varphi(a, c))$.

   Then there exists the set of all representatives $A^* \subset A$, i.e.,
   - $(\forall a \in A)(\exists b \in A^*)(\varphi(a, b))$,
   - $(\forall a, b \in A^*)(\varphi(a, b) \to a = b)$.

Proposition 5.2. ACA_0 proves Rees theorem for countable semigroups.

Proof.

- Take an element $a \in S$ and let $G \cong \{x \in S : x\mathcal{L}a \wedge x\mathcal{R}a\}$. This forms a group by Green's lemma (which is provable in RCA_0).
- By the previous lemma, let Λ, I be the sets of all representatives of left and right equivalence respectively.
- Take functions $r : I \to S$ such that $(\forall i \in I)(i\mathcal{R}r_i \wedge r_i\mathcal{L}a)$ and $q : \Lambda \to S$ such that $(\forall \lambda \in \Lambda)(\lambda\mathcal{L}q_\lambda \wedge q_\lambda\mathcal{R}a)$. Let $P_{\lambda i} = q_\lambda r_i$.

It follows that $S \cong M(G; I, \Lambda, P)$. □

6 Exploration for reversal

Lemma 6.1 (Simpson, [7])**.** The following are equivalent over RCA_0.

1. ACA_0.
2. For any injection $\alpha : \mathbb{N} \to \mathbb{N}$, there exists the image of α

$$\mathrm{Im}\alpha = \{j : (\exists i)(\alpha(i) = j)\}.$$

Proposition 6.2. The following is provable in RCA_0. Assume Rees theorem for countable semigroups. Then for any simple and complete semigroup S, the left equivalence of S exists.

Proof. $(i, g, \lambda), (j, h, \mu) \in M(G; I, \Lambda, P)$ are left equivalent if and only if $\lambda = \mu$. $\square$

To show that Rees theorem implies ACA, it is enough to construct simple and complete semigroup whose left equivalence encodes the image of given injection $\alpha : \mathbb{N} \to \mathbb{N}$.

Theorem 6.3. Let $\alpha : \mathbb{N} \to \mathbb{N}$ be an injection. In RCA_0 we can construct

1. a complete semigroup whose left equivalence encodes the image of α.
2. a simple semigroup whose left equivalence encodes the image of α.
3. a simple and complete *magma* M whose left equivalence encodes the image of α.

Remark 6.4. A set with a binary operation is said to be a *magma*. The binary operation need not to satisfy associativity. The notions of simplicity, completeness, and left equivalence can be extended to magmas naturally. Although the left equivalence of a magma need not to be equivalent relation.

Summary; partial results for reversal of Rees theorem

Finding a "semigroup" which encodes the image of given injection with...

simplicity	completeness	associativity	
yes	no	yes	✓
no	yes	yes	✓
yes	yes	no	✓
yes	yes	yes	WANTED

References

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