Kazuyuki Tanaka's work on AND-OR trees and subsequent development

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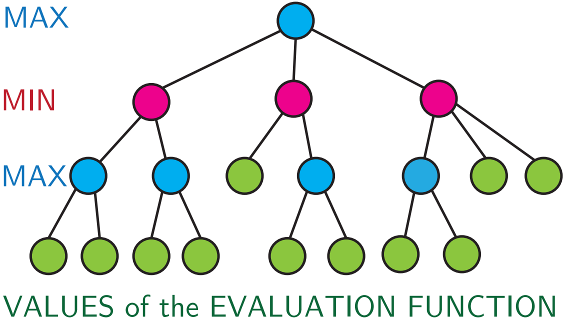
Abstract

Searching a game tree is an important subject of artificial intelligence. In the case where the evaluation function is bi-valued, the subject is interesting for logicians, because a game tree in this case is a Boolean function.

Kazuyuki Tanaka has a wide range of research interests which include complexity issues on AND-OR trees. In the joint paper with C.-G. Liu (2007) he studies distributional complexity of AND-OR trees. We overview this work and subsequent development.

Outline

Min-max search on a game tree

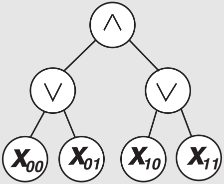


Time of computing $\simeq$ # of times of calling the evaluation function

Our setting

A uniform binary AND-OR tree T_2^k

T_2^1



- $\wedge$ = AND = Min.
- $\vee$ = OR = Max.
- T_2^{k+1} is defined by replacing each leaf of T_2^k with T_2^1.
- Find: root = 1 (TRUE) or 0 (FALSE)?
- Each leaf is hidden.
- Cost := # of leaves probed
- Allowed to skip a leaf (α-β pruning).

Alpha-beta pruning algorithm

Definition.

- Depth-first.
- A child of an AND-gate has the value 0

  $$\Downarrow$$

  Recognize that the AND-gate has the value 0
  without probing the other child (an alpha-cut).
- Similar rule applies to an OR-gate (a beta-cut).

Knuth, D.E. and Moore, R.W.: An analysis of alpha-beta pruning. *Artif. Intell.*, **6** pp. 293–326 (1975).

The optimality of alpha-beta pruning algo. for IID

- ID = independent distribution
- IID = independent and identical distribution
- CD = correlated distribution

In the case of IID:

The optimality of alpha-beta pruning algorithms is studied by Baudet (1978) and Pearl (1980), and the optimality is shown by Pearl (1982) and Tarsi (1983).

Baudet, G.M.: *Artif. Intell.*, 10 (1978) 173–199.
Pearl, J.: *Artif. Intell.*, **14** pp.113–138 (1980).
Pearl, J.: *Communications of the ACM*, 25 (1982) 559–564.
Tarsi, M.: *J. ACM*, **30** pp. 389–396 (1983).

A variant of von Neumann's min-max theorem

Yao's Principle (1977)

Randomized complexity *Distributional complexity*

$$\min_{A_R} \max_{\omega} \mathrm{cost}(A_R, \omega) = \max_{d} \min_{A_D} \mathrm{cost}(A_D, d),$$

ω : truth assignment

A_R : randomized algo.

A_D : deterministic algorithm

d : prob. distribution on the truth assignments

Yao, A.C.-C.: Probabilistic computations: towards a unified measure of complexity.
In: *Proc. 18th IEEE FOCS*, pp.222–227 (1977).

Estimation of the equilibrium value

Saks and Wigderson (1986)

For a perfect binary AND-OR tree,

$$(\text{Randomized Complexity}) \approx (\text{Constant}) \times \left(\frac{1+\sqrt{33}}{4}\right)^h,$$

where h is the height of the tree.

Saks, M. and Wigderson, A.:
Probabilistic Boolean decision threes and the complexity of evaluating game trees.
In: *Proc. 27th IEEE FOCS*, pp.29–38 (1986).

Kazuyuki Tanaka's work with C.-G. Liu (1)

Liu, C.-G. and Tanaka, K.:
Eigen-distribution on random assignments for game trees.
Inform. Process. Lett., **104** pp.73–77 (2007).

Preliminary versions:

In: *SAC '07* pp.78–79 (2007).

In: *AAIM 2007, LNCS* **4508** pp.241–250 (2007).

The eigen-distribution

Def.

"$\boldsymbol{d}$ is the *eigen-distribution* (for ▲▲▲)"
$\Leftrightarrow$ $\boldsymbol{d}$ has the property ▲▲▲ and

$$\min_{A_D} \mathrm{cost}(A_D, d) = \max_{\delta} \min_{A_D} \mathrm{cost}(A_D, \delta)$$

Here,
A_D runs over all deterministic alpha-beta pruning algorithms.
δ runs over all prob. distributions s.t. ▲▲▲.

▲▲▲ is e.g., ID.

They study the eigen-distributions in the two cases:
the ID-case and the CD-case.

The ID case

Theorem 4 (Liu and Tanaka, IPL (2007))

If d is the eigen-distribution for IDs then d is an IID.

Given k (i.e., height of $T_2^k = 2k$), define ϱ as follows.
The IID in which prob[the value is 0] $= \varrho$ at every leaf is the eigen-distribution among IID.

Theorem 5 (Liu and Tanaka, IPL (2007)) For T_2^k and IID:

- $$\frac{\sqrt{7}-1}{3} \leq \varrho \leq \frac{\sqrt{5}-1}{2}$$
- ϱ is strictly increasing function of k.

The CD case

Def. (Saks-Wigderson) The *reluctant inputs*

- When assigning 0 to an $\wedge$, assign 0 to exactly one child.
- When assigning 1 to an $\vee$, assign 1 to exactly one child.

Liu and Tanaka extend the above concept.

RAT (the reverse assignment technique), $i = 0, 1$

- i-set is the set of all reluctant inputs s.t. the root has value i.
- E_i-distribution is the dist. on i-set s.t.
  all the deterministic algorithm have the same complexity.

The CD case (continued)

Theorem 8 (Liu and Tanaka, IPL (2007)) For T_2^k:

E_i-distribution is the uniform distribution on i-set.

Theorem 9 (Liu and Tanaka, IPL (2007)) For T_2^k and CD:

E_1-distribution is the unique eigen-distribution.

Kazuyuki Tanaka's work with C.-G. Liu (2)

- Given a Boolean function f,
  $\beta(f)$ denotes the distributional complexity (i.e. the max-min-cost achieved by the eigen-distribution) w.r.t. 1-set.
- $\alpha(f)$ denotes that w.r.t. 0-set.
- Trees are not supposed to be binary in this paper.
- Given a tree T, they study recurrences on $\beta(f_T)$ and $\alpha(f_T)$.

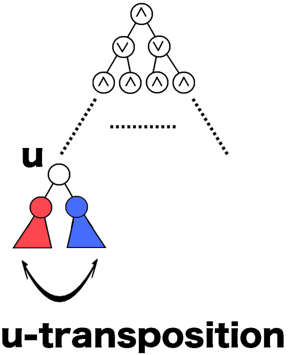
Liu, C.-G. and Tanaka, K.:
The computational complexity of game trees by eigen-distribution.
COCOA 2007, LNCS **4616** pp.323–334 (2007).

Extension (1): CD-case of T_2^k [1/5]

We consider classes of truth assignments (and, algorithms) closed under transpositions.
The concept of "a distribution achieving the equilibriumthe w.r.t. the given classes" is naturally defined.

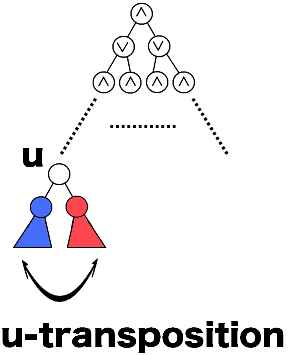
S. and Nakamura, R.:
The eigen distribution of an AND-OR tree under directional algorithms.
IAENG Internat. J. of Applied Math., **42**, pp.122-128 (2012).

Extension (1): CD-case of T_2^k [2/5]



of a node / a truth assignment / an algorithm

Extension (1): CD-case of T_2^k [2/5]



of a node / a truth assignment / an algorithm

Extension (1): CD-case of T_2^k [3/5]

Definition. Directional Algorithms

An alpha-beta pruning algorithm is said to be *directional* if for some linear ordering of the leaves it never selects for examination a leaf situated to the left of a previously examined leaf.

Suppose x, y and z are leaves.

- Allowed: To skip x.
- Not allowed:
  If (x is skipped) { scan y before z } else { scan z before y }

Pearl, J.: Asymptotic properties of minimax trees and game-searching procedures. *Artif. Intell.*, **14** pp.113–138 (1980).

Extension (1): CD-case of T_2^k [4/5]

S. and Nakamura (2012)

(a) The Failure of the Uniqueness

In the situation where only directional algorithms are considered, the uniqueness of $\boldsymbol{d}$ achieving the equilibrium fails.

(b) A Counterpart of the Liu-Tanaka Theorem

In the situation where only directional algorithms are considered, A weak version of the Liu-Tanaka theorem holds. (1) is equivalent to (2).

(1) $\boldsymbol{d}$ achieves the equilibrium.

(2) $\boldsymbol{d}$ is on the 1-set and the cost does not depend on an algorithm.

Extension (1): CD-case of T_2^k [5/5]

A key to the result (b) is the following.

No-Free-Lunch Theorem (Wolpert and Macready, 1995)

(Under certain assumptions)
Averaged over all cost functions,
all search algorithms give the same performance.

Wolpert, D.H. and MacReady, W.G.:
No-free-lunch theorems for search,
Technical report SFI-TR-95-02-010, Santa Fe Institute, Santa Fe, New Mexico (1995).

Extension (2): ID-case of T_2^k [1/4]

Theorem 4 (Liu and Tanaka, 2007)

If d achieves the equilibrium among IDs then d is an IID.

Their proof: "It is not hard."

Is it (↑) really easy to prove? No. A brutal induction does not work. We show a stronger form of Theorem 4 with clever tricks of induction.

S. and Niida, Y.:
Equilibrium points of an AND-OR tree: Under constraints on probability,
Ann. Pure Appl. Logic , 166, pp. 1150–1164 (2015).

Extension (2): ID-case of T_2^k [2/4]

Keys to the solution

Lemma 1 (S. and Niida, 2015)

Consider an IID on an OR-AND tree.
$x :=$ prob. of a leaf (having the value 0).
$p(x) :=$ prob. of the root (having the value 0).
$c(x) :=$ expected cost of the root.

Then, both of the followings are decreasing functions of x $(0 < x < 1)$.

$$\frac{c(x)}{p(x)}, \qquad \frac{c'(x)}{p'(x)}$$

Extension (2): ID-case of T_2^k [3/4]

Lemma 2 (S. and Niida, 2015)

A certain constraint extremum problem has a unique solution.

The proof highlight: By means of Lemma 1, the objective function is decreasing in a certain open interval. $\square$

Remark:
At the maximizer, the objective function is NOT differentiable.

Extension (2): ID-case of T_2^k [4/4]

Theorem (S. and Niida, 2015)

Fix an r ($0 < r < 1$). Let rID denote an ID s.t. prob. of the root (having the value 0) is r.

If d achieves the equilibrium among rIDs then d is an IID.

As a corollary

Theorem 4 (Liu and Tanaka, 2007)

If d achieves the equilibrium among IDs then d is an IID.

Extension (3): ID-case for more general trees

Recently, NingNing Peng, Yue Yang , Keng Meng Ng and Kazuyuki Tanaka extend the results of S. and Niida (2015) to trees not necessarily binary.

Thank you for your attention.

Happy 60th birthday.

- Baudet, G.M.: On the branching factor of the alpha-beta pruning algorithm, *Artif. Intell.*, 10 (1978) 173–199.
- Knuth, D.E. and Moore, R.W.: An analysis of alpha-beta pruning. *Artif. Intell.*, **6** pp. 293–326 (1975).
- Liu, C.-G. and Tanaka, K.: Eigen-distribution on random assignments for game trees. *Inform. Process. Lett.*, **104** pp.73–77 (2007).
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Saks, M. and Wigderson, A.: Probabilistic Boolean decision threes and the complexity of evaluating game trees. In: *Proc. 27th IEEE FOCS*, pp.29–38 (1986).

Suzuki, T. and Nakamura, R.: The eigen distribution of an AND-OR tree under directional algorithms. *IAENG Internat. J. of Applied Math.*, **42**, pp.122-128 (2012). `www.iaeng.org/IJAM/issues_v42/issue_2/index.html`

Suzuki, T. and Niida, Y.: Equilibrium points of an AND-OR tree: Under constraints on probability, *Ann. Pure Appl. Logic* , 166, pp. 1150–1164 (2015).

Tarsi, M.: Optimal search on some game trees. *J. ACM*, **30** pp. 389–396 (1983).

Wolpert, D.H. and MacReady, W.G.: No-free-lunch theorems for search, *Technical report SFI-TR-95-02-010*, Santa Fe Institute, Santa Fe, New Mexico (1995).

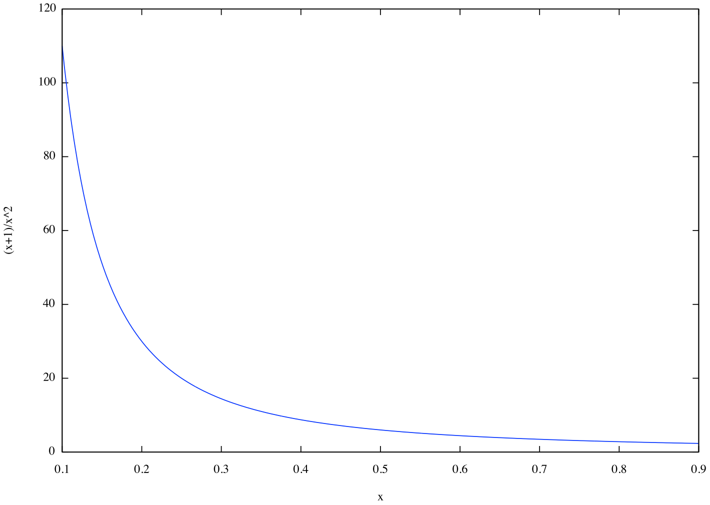

Figure 0: $c_{\vee,1}(x)/p_{\vee,1}(x)$ $(0.1 < x < 0.9)$

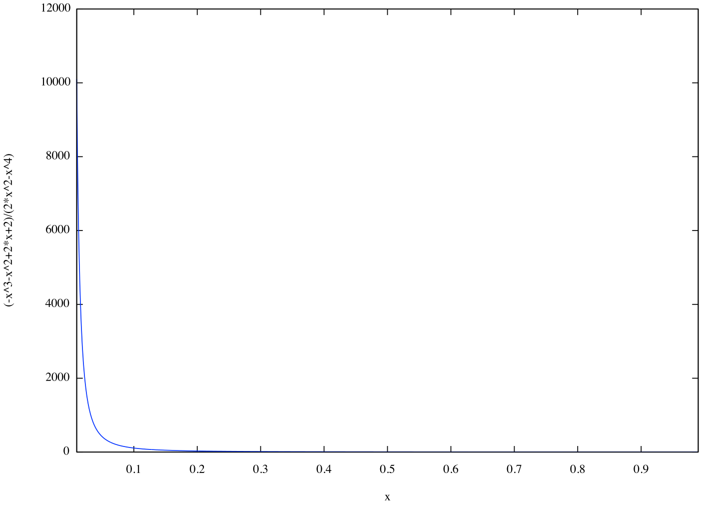

Figure 1: $c_{\vee,2}(x)/p_{\vee,2}(x)$ $(0 < x < 1)$

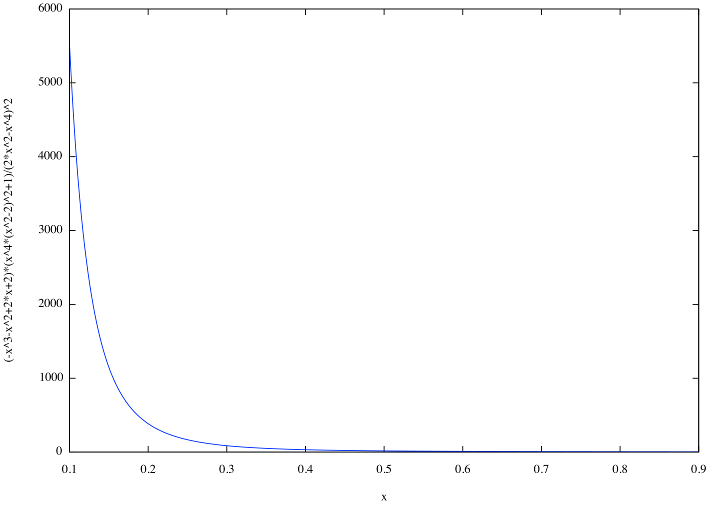

Figure 2: $c_{\vee,3}(x)/p_{\vee,3}(x)$ $(0.1 < x < 0.9)$

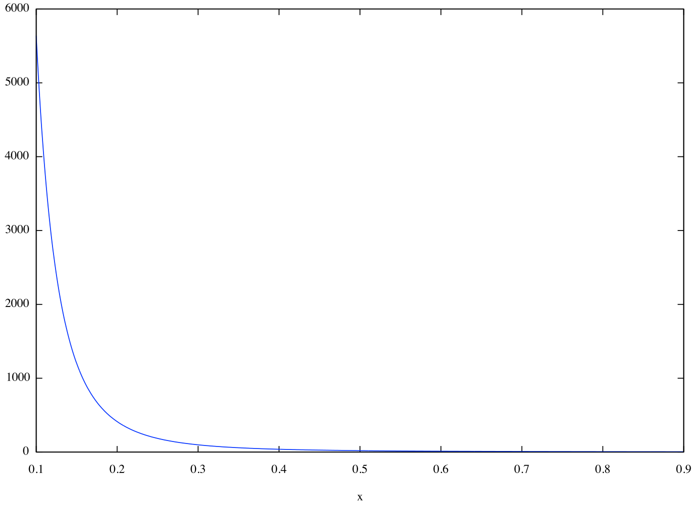

Figure 3: $c_{\vee,4}(x)/p_{\vee,4}(x)$ $(0.1 < x < 0.9)$

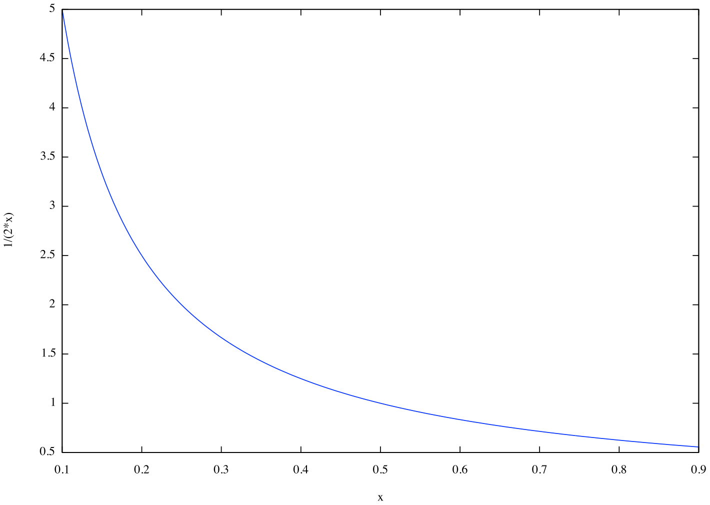

Figure 4: $c'_{\vee,1}(x)/p'_{\vee,1}(x)$ $(0.1 < x < 0.9)$

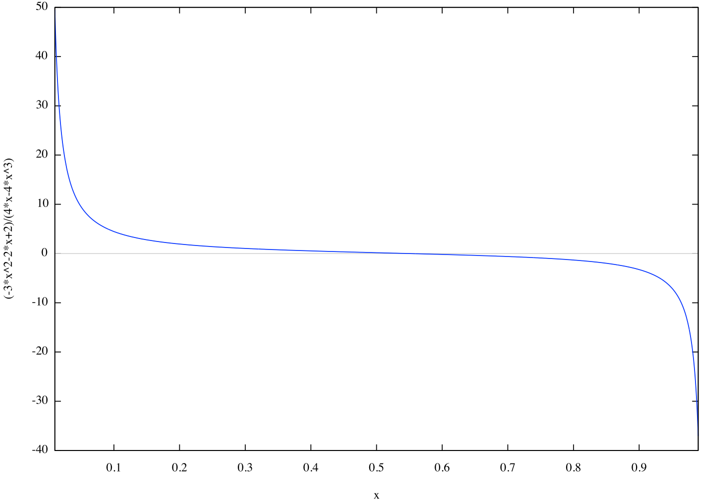

Figure 5: $c'_{\vee,2}(x)/p'_{\vee,2}(x)$ $(0 < x < 1)$

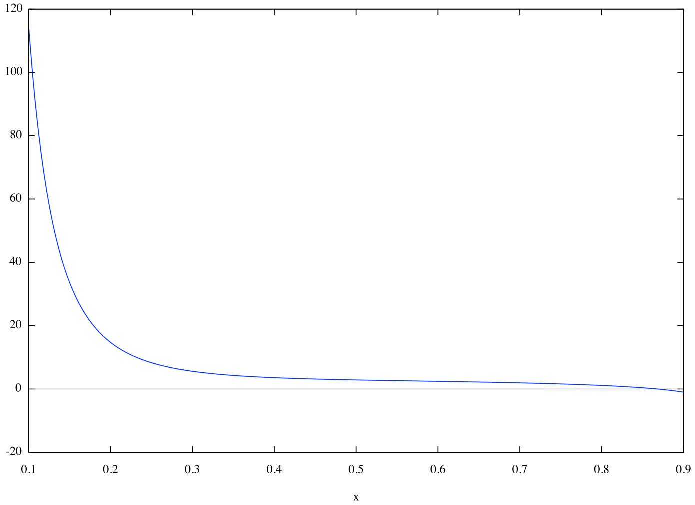

Figure 6: $c'_{\vee,3}(x)/p'_{\vee,3}(x)\ (0.1 < x < 0.9)$

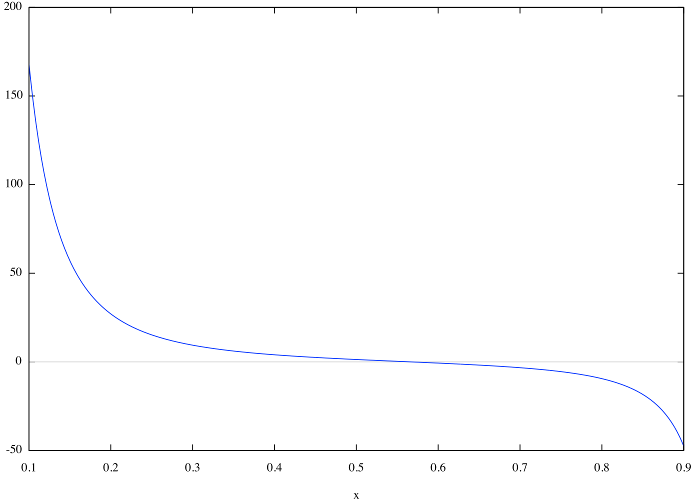

Figure 7: $c'_{\vee,4}(x)/p'_{\vee,4}(x)$ $(0.1 < x < 0.9)$