Effective Reducibility for Smooth and Analytic Equivalence Relations on a Cone

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Joint Work with

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1. Invariant descriptive set theory:

2. Computable structure theory:

1. Invariant descriptive set theory: classification of classification problems of mathematical structures such as:
   - Isomorphism relation on countable Boolean algebras.
   - Isomorphism relation on countable p-groups.
   - Isometry relation on Polish metric spaces.
   - Linear isometry relation on separable Banach spaces.
   - Isomorphism relation on separable C^*-algebras.

   Key notion: Borel reducibility among equivalence relations on Borel spaces.
2. Computable structure theory:

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   Key notion: Borel reducibility among equivalence relations on Borel spaces.
2. Computable structure theory: classification of classification problems of *computable structures* such as:
   - Isomorphism relation of computable trees.
   - Isomorphism relation of computable torsion-free abelian grps
   - Bi-embeddability relation of computable linear orders.

   Key notion: computable reducibility among equivalence relations on *represented spaces*.

- (X, δ) is a *represented space* if $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ is a partial surjection.
- A point $x \in X$ is *computable* if it has a computable name,
  that is, there is a computable $p \in \delta^{-1}\{x\}$.

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Example

1. The *space of countable $\mathcal{L}$-structures* is represented:
   For a countable relational language $\mathcal{L} = (R_i)_{i \in \mathbb{N}}$,
   each countable $\mathcal{L}$-structure K with domain $\subseteq \omega$
   is identified with its atomic diagram $D(K) = \oplus_{i \in \mathbb{N}} R_i^K \in 2^\omega$.
   For a class $\mathbb{K}$ of countable $\mathcal{L}$-structures with $\delta : D(K) \mapsto K$,
   $(\mathbb{K}, \delta)$ forms a represented space.
2. Polish spaces, second-countable T_0 space are represented.
3. Much more generally, every T_0 space with a countable cs-network has a "universal" representation δ, i.e., for any representation δ', there is a continuous map g such that $\delta' = \delta \circ g$.

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- A point $x \in X$ is *computable* if it has a computable name, that is, there is a computable $p \in \delta^{-1}\{x\}$.
- The e-th computable point of $\mathcal{X} = (X, \delta)$ is denoted by $\Phi_e^{\mathcal{X}}$.

Let E and F be equivalence relations on represented spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. We say that $E \leq_{\mathrm{eff}} F$ if there is a partial computable function $f :\subseteq \mathbb{N} \to \mathbb{N}$ such that for all $i, j \in \mathbb{N}$ with $\Phi_i^{\mathcal{X}}, \Phi_j^{\mathcal{X}} \in \mathbf{dom}(\delta_X)$,

$$\Phi_i^{\mathcal{X}} E \Phi_j^{\mathcal{X}} \iff \Phi_{f(i)}^{\mathcal{Y}} F \Phi_{f(j)}^{\mathcal{Y}}.$$

Let E and F be equivalence relations on Borel spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. We say that $E \leq_{\mathrm{B}} F$ if there is a Borel function $f : \mathcal{X} \to \mathcal{Y}$ such that for all $x, y \in \mathcal{X}$,

$$xEy \iff f(x)Ff(y).$$

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"Effective reducibility on a cone"

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  - (K.-Pauly) Turing degree spectrum / Scott ideals $\rightsquigarrow$ a construction of linearly non-isometric (ring non-isomorphic, etc.) examples of Banach algebras of real-valued Baire n functions on Polish spaces.

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$$\Phi^{z,\mathcal{X}}_i E \Phi^{z,\mathcal{X}}_j \iff \Phi^{z,\mathcal{Y}}_{f(i)} F \Phi^{z,\mathcal{Y}}_{f(j)}.$$

$$\begin{array}{ccc} E \leq_{\mathrm{c}} F & \Longrightarrow & E \leq_{\mathrm{B}} F \\ \Downarrow & & \Downarrow \\ E \leq^{\mathrm{cone}}_{\mathrm{eff}} F & \Longrightarrow & E \leq^{\mathrm{cone}}_{\mathrm{hyp}} F \end{array}$$

- E is said to be *analytic $\leq^{\mathrm{cone}}_{\mathrm{eff}}$-complete* if $F \leq^{\mathrm{cone}}_{\mathrm{eff}} E$ for any analytic equivalence relation F.
- E is said to be *$\leq^{\mathrm{cone}}_{\mathrm{eff}}$-intermediate* if
  - E is not analytic $\leq^{\mathrm{cone}}_{\mathrm{eff}}$-complete,
  - and there is no Borel eq. relation F such that $E \leq^{\mathrm{cone}}_{\mathrm{eff}} F$.

The Vaught Conjecture (1961)

The number of countable models of a first-order theory is at most countable or $\mathbf{2}^{\aleph_0}$.

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- (Topological Vaught conjecture) The number of orbits of a continuous action of a Polish group on a standard Borel space is at most countable or $\mathbf{2}^{\aleph_0}$.

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Fact (Becker 2013; Knight and Montalbán)

Suppose that there is no $\mathcal{L}_{\omega_1\omega}$-axiomatizable class of countable structures whose isomorphism relation is $\leq^{\mathbf{cone}}_{\mathbf{eff}}$-intermediate then, the $\mathcal{L}_{\omega_1\omega}$-Vaught conjecture is true.

Indeed, if there is no $\leq^{\mathbf{cone}}_{\mathbf{eff}}$-intermediate orbit equivalence relation then, the topological Vaught conjecture is true.

The differences of $\leq_{\mathbf{B}}$ and $\leq^{\mathbf{cone}}_{\mathbf{eff}}$ among non-Borel orbit eq. relations:
For Borel reducibility (H. Friedman and Stenley 1989):

- The isomorphism relation on an $\mathcal{L}_{\omega_1\omega}$-axiomatizable class of countable structure *CANNOT* be analytic $\leq_{\mathbf{B}}$*-complete*.
- Moreover, the isomorphism relation on countable torsion abelian groups is *NOT* $\leq_{\mathbf{B}}$*-complete* even among isomorphism relations on classes of countable structures.

For computable reducibility (Fokina, S. Friedman, et al. 2012):

- The isomorphism relations on computable graphs, torsion-free abelian groups, fields (of a fixed characteristic), etc. are $\leq_{\mathbf{eff}}$*-complete* analytic equivalence relations.
- The isomorphism relation on computable torsion abelian groups is also a $\leq_{\mathbf{eff}}$*-complete* analytic equivalence relation.

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In this talk, we focus on the differences of $\leq_{\mathbf{B}}$ and $\leq^{\mathbf{cone}}_{\mathbf{eff}}$ among

- non-Borel non-orbit analytic equivalence relations,
- and smooth equivalence relations.

Non-orbit analytic equivalence relations:

$xE_{\mathrm{wo}}y :\iff$ either $x, y \notin \mathbf{WO}$ or x and y are isomorphic as w.o.
$xE_{\mathrm{ck}}y :\iff \omega_1^x = \omega_1^y$ holds.

Fact

- (Gao) E_{wo} and E_{ck} are $\leq_{\mathbf{B}}$-incomparable.
- (Coskey-Hamkins 2011) E_{wo} and E_{ck} are $\leq_{\mathrm{ITTM}}$-bireducible.

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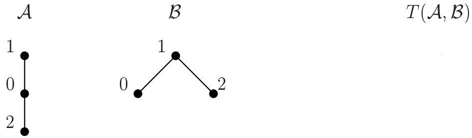
Conjecture

If $x^\sharp$ exists for any real x, then $E_{\mathrm{ck}} \equiv^{\mathrm{cone}}_{\mathrm{eff}} E_{\mathrm{wo}}$.

$T(\mathcal{A},\mathcal{B})$: the tree of partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$.



For partial orders $\mathcal{A} = (A, \leq_A)$ and $\mathcal{B} = (B, \leq_B)$ with $A, B \subseteq \omega$
$\sigma \oplus \tau \in T(\mathcal{A},\mathcal{B})$ iff

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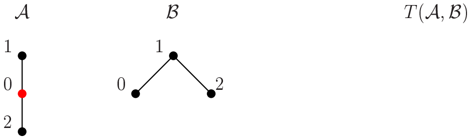

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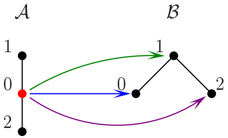

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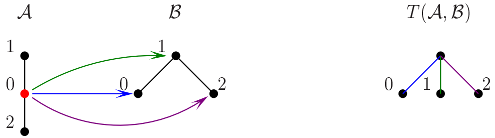

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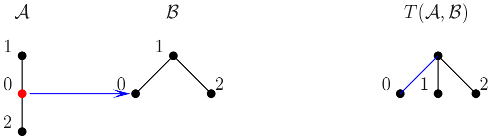

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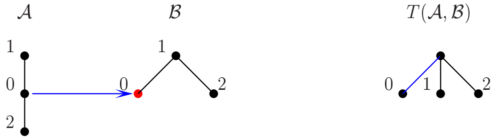

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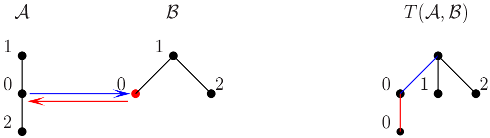

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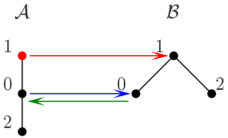

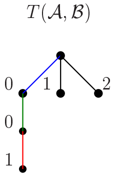

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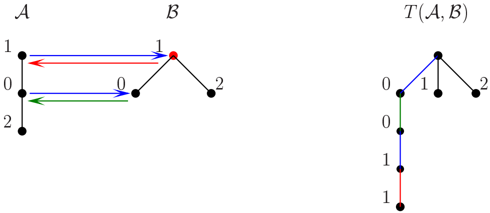

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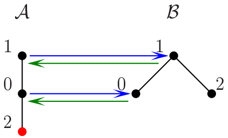

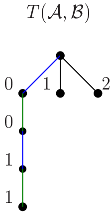

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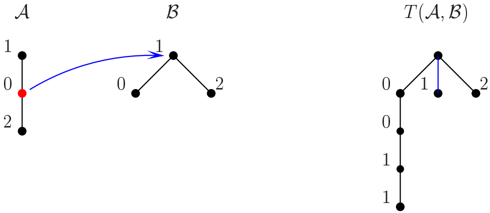

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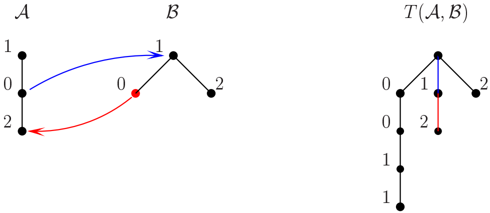

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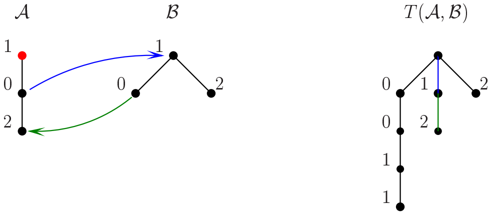

For partial orders $\mathcal{A} = (A, \leq_A)$ and $\mathcal{B} = (B, \leq_B)$ with $A, B \subseteq \omega$
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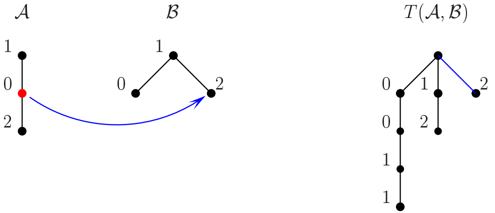

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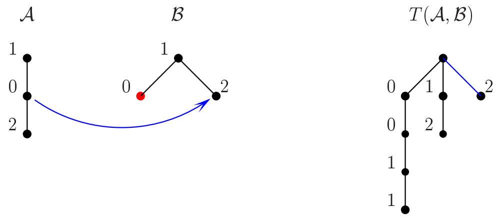

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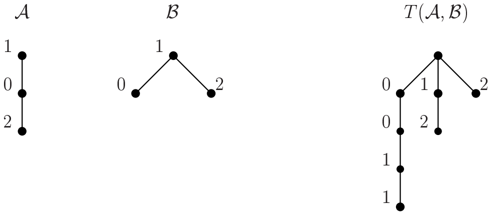

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$T(\mathcal{A}, \mathcal{B})$: the tree of partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$.

Lemma (Upper Bound)

$\alpha < \beta < \omega_1$: ordinals.
$\mathcal{A} \in \mathbf{LO}$ s.t. $\mathbf{otype}(\mathcal{A}) = \alpha + \lambda$, where λ has no least element.
$\mathcal{B} \in \mathbf{LO}$ s.t. $\mathbf{otype}(\mathcal{B}) = \beta + \theta$ for a linear order θ.
Then, $\mathbf{rank}(T(\mathcal{A}, \mathcal{B})) \leq \omega^{\alpha+2}$.

β is *α-closed* if $(\forall \gamma < \alpha)(\forall \delta < \beta)\ \delta + \gamma < \beta$.

Lemma (Lower Bound)

$\alpha, \beta < \omega_1$: ordinals, β is ω^α-closed, $c \in \omega$
$\mathcal{A} \in \mathbf{WO}$ s.t. $\mathbf{otype}(\mathcal{A}) = \omega^\alpha \cdot c$.
$\mathcal{B} \in \mathbf{WO}$ s.t. $\mathbf{otype}(\mathcal{B}) = \beta$.
Then, $\mathbf{rank}(T(\mathcal{A}, \mathcal{B})) \geq \omega \cdot \alpha$.

Lemma

$\mathcal{A}$: a well order s.t. $\mathbf{otype}(\mathcal{A}) = \alpha$.
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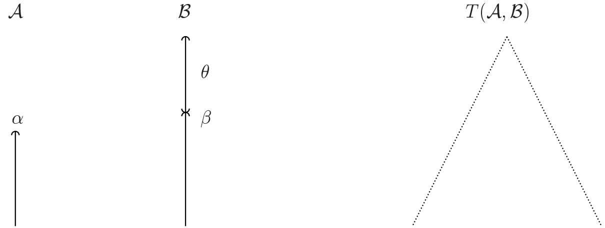

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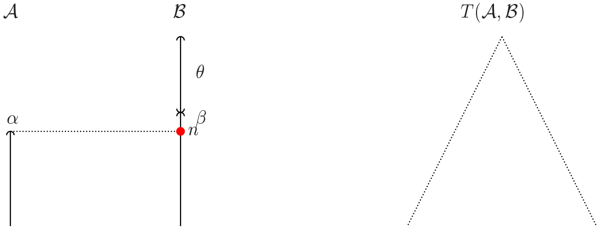

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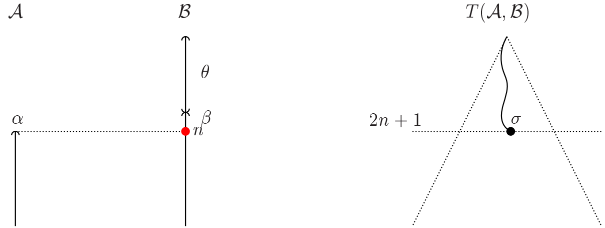

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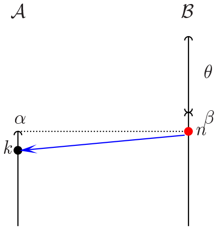

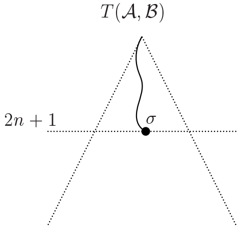

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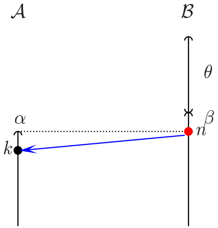

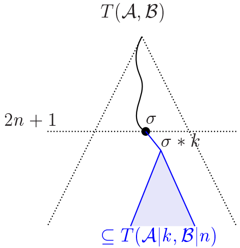

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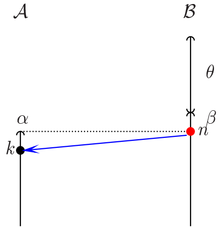

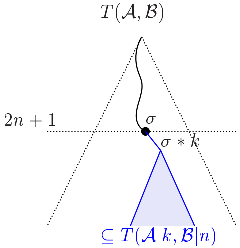

$\mathbf{rank}(T(\mathcal{A}, \mathcal{B})) \leq \sup_k \mathrm{rank} T(\mathcal{A} \upharpoonright k, \mathcal{B} \upharpoonright n) + 2n + 1$.

Lemma (Upper Bound)

$\mathcal{A} \in \mathbf{LO}$ s.t. $\mathbf{otype}(\mathcal{A}) = \alpha + \lambda$, where λ has no least element.
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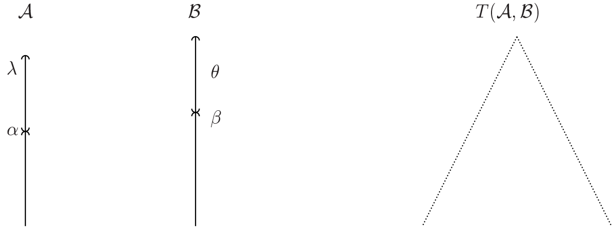

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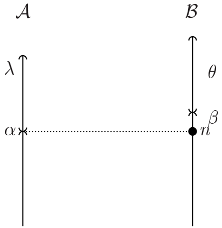

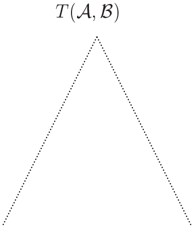

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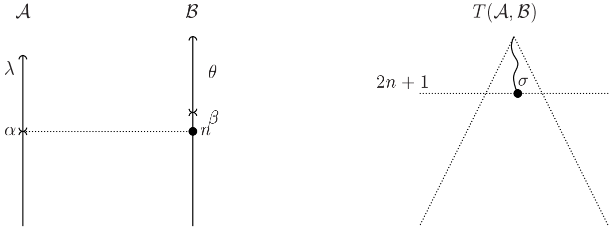

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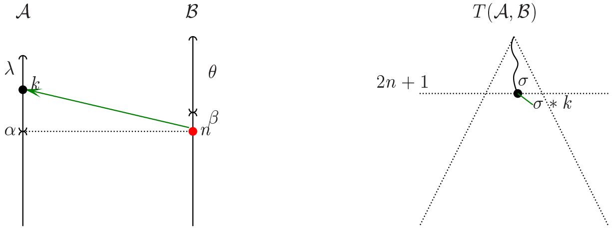

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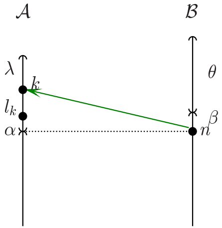

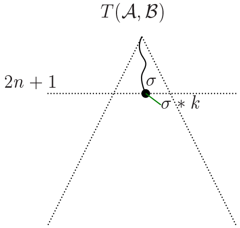

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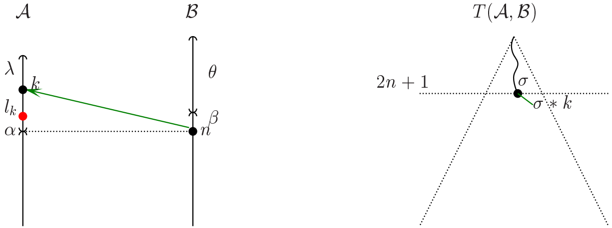

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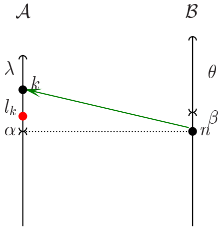

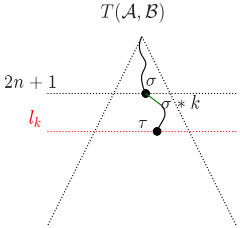

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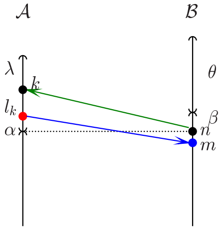

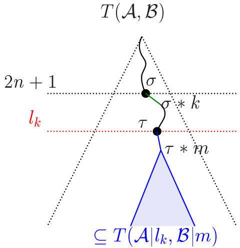

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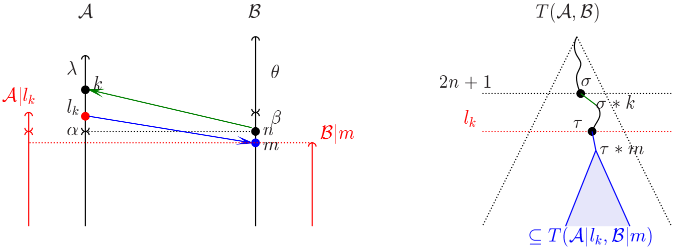

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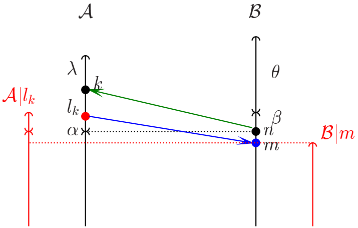

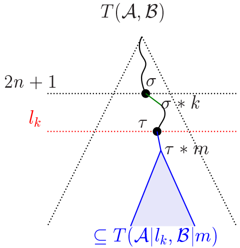

- $\mathbf{rank}(T(\mathcal{A}, \mathcal{B})) \leq \sup_k(\sup_m \mathbf{rank}T(\mathcal{A} \upharpoonright l_k, \mathcal{B} \upharpoonright m)_{+l_k})_{+2n+1}$.
- $\mathbf{rank}(\mathcal{A} \upharpoonright l_k, \mathcal{B} \upharpoonright m) \leq \omega^{\alpha_m+1}$, where $\alpha_m := \mathbf{otype}(\mathcal{B} \upharpoonright m) < \alpha$.

$\mathcal{A} \in \mathbf{WO}$ s.t. $\mathbf{otype}(\mathcal{A}) = \omega^\alpha$.
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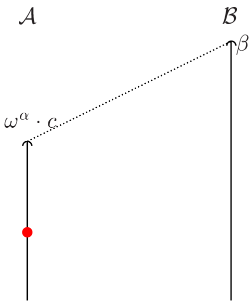

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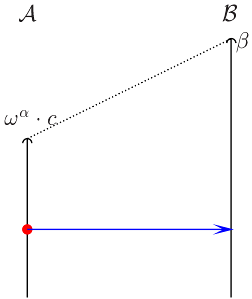

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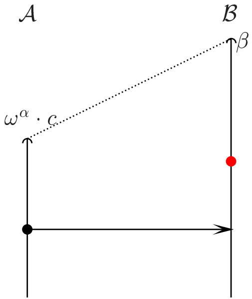

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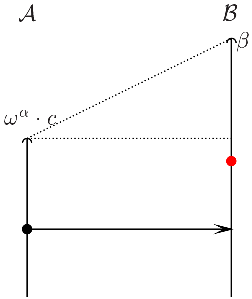

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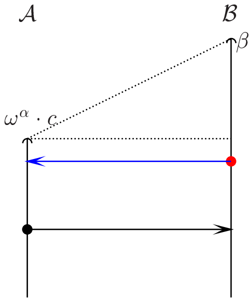

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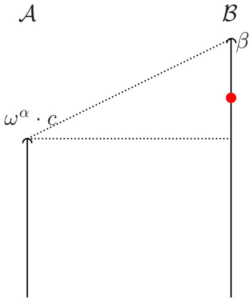

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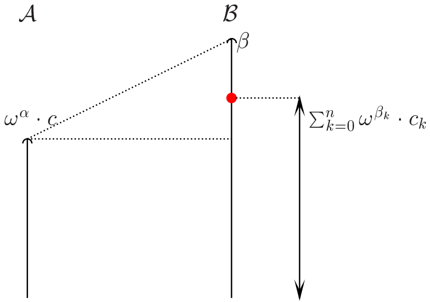

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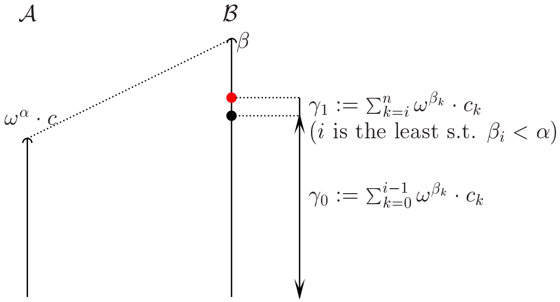

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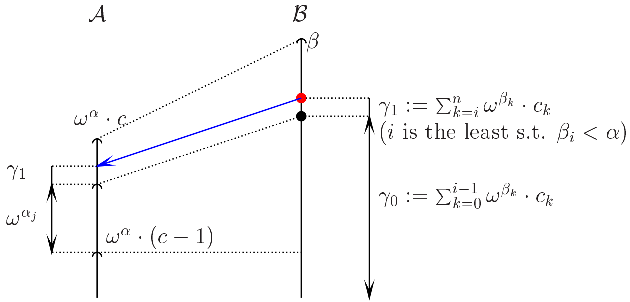

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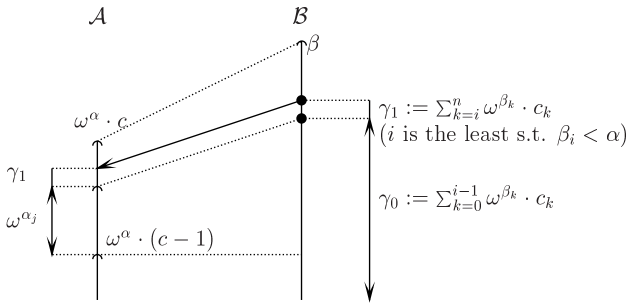

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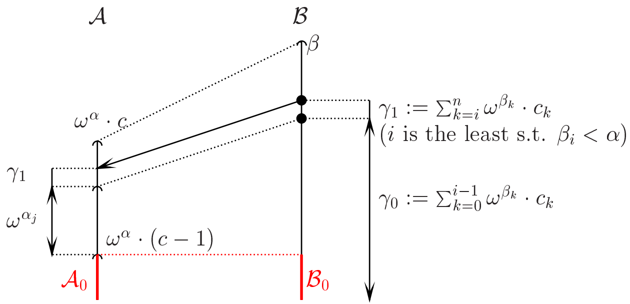

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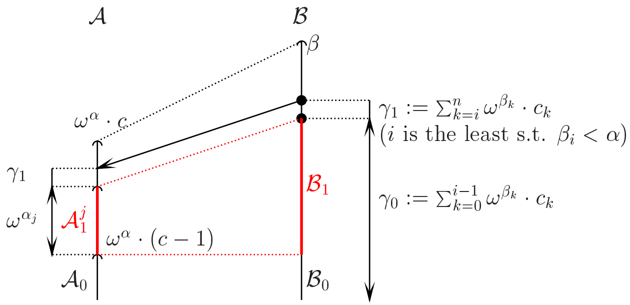

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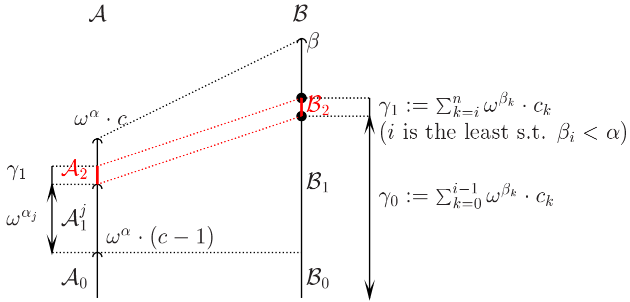

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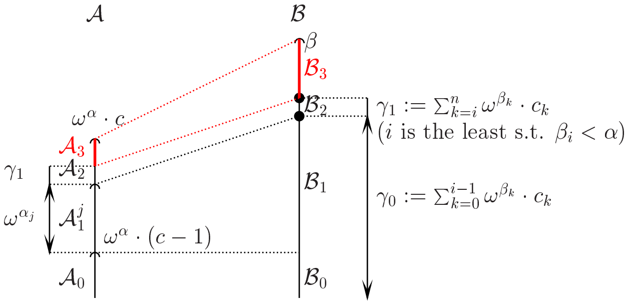

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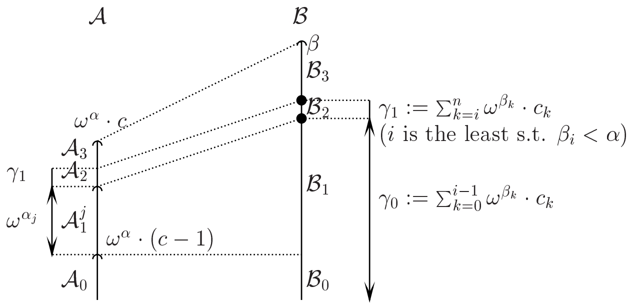

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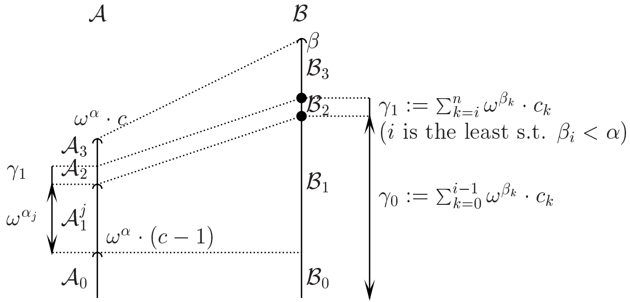

- If α is limit, choose an increasing seq. $\alpha_0 < \alpha_1 < \cdots \to \alpha$.
- If α is successor, we use $\omega^{(\alpha-1)} \cdot j$ instead of ω^{α_j}.
- $\mathcal{A}_0 \simeq \mathcal{B}_0 \simeq \omega^{\alpha} \cdot (c-1)$ and $\mathcal{A}_2 \simeq \mathcal{B}_2 \simeq \gamma_1$.
- $\mathcal{A}_1^j \simeq \omega^{\alpha_j}$, $\mathcal{B}_1 \simeq \gamma_0$; $\mathcal{A}_3 \simeq \omega^{\alpha}$, $\mathcal{B}_3$ is ω^{α}-closed.

$(L, <_L)$: a linear order
Define the linear order $\omega^L = (\mathrm{CNF}(L), \leq_{\omega^L})$ as follows:

1. $\mathrm{CNF}(L) = \{(\lambda_i, c_i)_{i<n} \in (L \times \omega)^{<\omega} : (\forall i)\ \lambda_{i+1} <_L \lambda_i\}$,
2. $(\lambda_i, c_i)_{i<n} \leq_{\omega^L} (\lambda'_j, c'_j)_{j<m} \iff (\exists k < m, n)$ s.t.
   - $(\forall i < k)\ \lambda_i = \lambda'_i$ and
   - $\lambda_k <_L \lambda'_k$ or $(\lambda_k = \lambda'_k$ and $c_i \leq c^*_i)$.

- Inductively define $\mathbf{exp}^0(L) = L$ and $\mathbf{exp}^{n+1}(L) = \omega^{\mathbf{exp}^n(L)}$.
- Define $\varepsilon(L)$ by $\sum_{n\in\omega} \mathbf{exp}^n(L)$.

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- If L is not well-ordered, then so is ω^L.
- $L \in \mathrm{WO}$, $(\lambda_i, c_i)_{i<n} \approx \sum_{i<n} \omega^{\lambda_i} \cdot c_i$.
- $L \in \mathrm{WO}$, $\mathrm{otype}(L) = \alpha \Rightarrow \mathrm{otype}(\omega^L) = \omega^\alpha$.

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Proof of $E_{\mathrm{ck}} \leq_{\mathrm{eff}}^{\mathrm{cone}} E_{\mathrm{wo}}$

1. $\mathcal{H}^x$: Harrison's pseudo well order relative to x whose order type is $\omega_1^x \cdot (1+\eta)$.
2. Given z and $x \leq_T z$, define $f(x) := \varepsilon(\mathrm{KB}(T(\mathcal{H}^x, \mathcal{H}^z)))$.

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4. If $\omega_1^x < \omega_1^z$, $\omega \cdot \omega_1^x \leq \mathrm{rank}(T(\mathcal{H}^x, \mathcal{H}^z)) \leq \omega^{\omega_1^x + 2}$.
   - $\varepsilon(\omega \cdot \omega_1^x)$ is isomorphic to $\varepsilon(\omega^{\omega^{\omega_1^x + 2}})$.
   - Hence, $\mathrm{otype}(\varepsilon(\mathrm{KB}(T(\mathcal{H}^x, \mathcal{H}^z)))) = \varepsilon(\omega_1^x)$.
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   - Thus, $\omega_1^x = \omega_1^y < \omega_1^z$ implies $f(x) \approx f(y) \approx \varepsilon(\omega_1^x) = \varepsilon(\omega_1^y)$.
5. Thus, $\omega_1^x = \omega_1^y \iff f(x), f(y) \notin \mathrm{WO}$ or $f(x) \approx f(y)$.

Proof of "$V = L$ implies $E_{ck} <^{\text{cone}}_{\text{eff}} E_{wo}$"

- Weitkamp (1982): If V is a generic extension of L, then the following set contains no Turing cone:

$$\{x \in 2^\omega : \omega_1^x \text{ is a recursively inaccessible ordinal}\}.$$

- Given r, choose $z \geq_T r$ s.t. ω_1^z is NOT rec. inaccessible.

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- Thus, there is no z-effective reduction from E_{wo} to E_{ck} since $\{x \leq z : x \notin \mathrm{WO}\}$ is $\Sigma^1_1(z)$-complete.

Non-orbit analytic equivalence relations:

$xE_{\mathrm{wo}}y :\iff$ either $x, y \notin \mathbf{WO}$ or x and y are isomorphic as w.o.
$xE_{\mathrm{ck}}y :\iff \omega_1^x = \omega_1^y$ holds.

Fact

- (Gao) E_{wo} and E_{ck} are $\leq_{\mathbf{B}}$-incomparable.
- (Coskey-Hamkins 2011) E_{wo} and E_{ck} are $\leq_{\mathrm{ITTM}}$-bireducible.

Theorem

- $E_{\mathrm{ck}} \leq^{\mathbf{cone}}_{\mathbf{eff}} E_{\mathrm{wo}}$.
- If $V = L$, then $E_{\mathrm{ck}} <^{\mathbf{cone}}_{\mathbf{eff}} E_{\mathrm{wo}}$.

Conjecture

If $x^\sharp$ exists for any real x, then $E_{\mathrm{ck}} \equiv^{\mathbf{cone}}_{\mathbf{eff}} E_{\mathrm{wo}}$.

Smooth Equivalence Relations

Δ_X: the equality $(X, =)$ on a topological space X.

$\leq_B$ ($\leq_c$, resp.): Borel (continuous, resp.) reducibility.

1. $\Delta_X \equiv_B \Delta_Y$ whenever X and Y are uncountable standard Borel spaces. In particular, $\Delta_{2^\omega} \equiv_B \Delta_{I^n} \equiv_B \Delta_{I^\omega}$
2. $\Delta_{2^\omega} <_c \Delta_I <_c \Delta_{I^2} <_c \cdots <_c \cdots <_c \Delta_{I^n} <_c \Delta_{I^{n+1}} < \Delta_{I^\omega}$.

Theorem

1. $\Delta_{2^\omega} <^{\text{cone}}_{\text{eff}} \Delta_I <^{\text{cone}}_{\text{eff}} \Delta_{I^2}$.
2. $\Delta_{I^3} \equiv^{\text{cone}}_{\text{eff}} \Delta_{I^4} \equiv^{\text{cone}}_{\text{eff}} \cdots \equiv^{\text{cone}}_{\text{eff}} \Delta_{I^n} \equiv^{\text{cone}}_{\text{eff}} \Delta_{I^{n+1}} \equiv^{\text{cone}}_{\text{eff}} \Delta_{I^\omega}$.

Remark

- $\mathbf{\Delta}_X \leq_{\mathrm{eff}} \mathbf{\Delta}_Y$ iff $\exists$ a Markov computable injection $f : X_{\mathrm{cpt}} \to Y_{\mathrm{cpt}}$.
- (Kreisel-Lacombe-Shoenfield) $f : (\omega^\omega)_{\mathrm{cpt}} \to (\omega^\omega)_{\mathrm{cpt}}$ is Markov computable iff it is computable in the sense of TTE.
- (de Brecht) X has a total admissible representation iff X is quasi-Polish.
- Hence, whenever X and Y are quasi-Polish, $\mathbf{\Delta}_X \leq_{\mathrm{eff}} \mathbf{\Delta}_Y$ iff there is a TTE-computable injection $f : X_{\mathrm{cpt}} \to Y_{\mathrm{cpt}}$.

Proof Idea of $\Delta_{I^n} \leq^{\text{cone}}_{\text{eff}} \Delta_{I^3}$

1. The n-dimensional sphere S^n is *not* an absolute extensor for I^{n+1}.
2. S^n is an absolute extensor for a normal space X
   $\iff$ the covering dimension of X is at most n.
3. If the covering dimension of a separable metric space X is $\leq n$, then it is embedded into the n-dimensional Nöbeling space $N^n \subseteq I^{2n+1}$.

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   Hence, I^n_{cpt} is computably embedded into $N^1 \subseteq I^3$.