# Reverse Mathematics and Whitehead Groups

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## Acknowledgement

This is a joint work with

- Frank STEPHAN (NUS)
- YANG Sen (Inner Mongolia University, China)
- YU Liang (Nanjing University, China)

# Outline

# Free groups

In this talk, groups are all *abelian*.

Let $F$ be a group.

- $B \subset F$ generates $F$
- $B \subset F$ is independent
- $B \subset F$ is a basis
- $F$ is free, if it has a basis $B$, e.g., $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$ etc.

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# Whitehead Groups

## Definition

Given groups $F$ and $G$, surjective homomorphism $\pi : G \to F$, we say that $\rho : F \to G$ is a splitting of $\pi$ if

- $\rho$ is a homomorphism.
- $\pi\rho = id_F$.

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# Whitehead Problem

**Lemma**

*Every free group is a W-group.*

**Whitehead's problem**: Is every Whitehead group free?

**Theorem (Stein 1951)**

*Every countable Whitehead group is free.*

**Theorem (Shelah 1974)**

*Whitehead Problem is independent of ZFC.*

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# A Quote from Wikipedia

*Shelah's result was completely unexpected. While the existence of undecidable statements had been known since Gödel's incompleteness theorem of 1931, previous examples of undecidable statements (such as the continuum hypothesis) had all been in pure set theory. The Whitehead problem was the first purely algebraic problem to be proved undecidable.*

# Introducing Reverse Mathematics

- Shelah's result separated Whitehead group and free group.
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- Will working within the second order arithmetic offer us a clearer picture?

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# Whitehead's problem in $RCA_0$

- Concepts like "Abelian group", "basis of a group", "free group", "splitting of a homomorphism", "$\mathbb{Z}$", "Whitehead group" are all expressible in second order arithmetic.
- Whitehead's problem can be formulated within second order arithmetic.
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# Basic Results about Freedom and Whitehead

In $RCA_0$,

- Every subgroup of a free group is free.
- Every subgroup of a W group is W.
- A free group is torsion free.
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# Results reported in NUS

**Theorem**

*In* $\mathrm{ACA}_0$*, Stein's theorem holds, i.e. every Whitehead group* $G$ *is free.*

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*Over* $\mathrm{WKL}_0$*, Stein's theorem implies* $\mathrm{ACA}_0$*. Hence* $\mathrm{WKL}_0 \vdash$ *Stein's Theorem* $\Leftrightarrow \mathrm{ACA}_0$*.*

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### Theorem

*Over* $\mathrm{WKL}_0$*, Stein's theorem implies* $\mathrm{ACA}_0$*. Hence* $\mathrm{WKL}_0 \vdash$ *Stein's Theorem* $\Leftrightarrow \mathrm{ACA}_0$*.*

# Over the base theory $\mathsf{RCA}_0$

**Theorem**

*Let REC be the minimal model of* $\mathsf{RCA}_0$*. Then*

$$\mathit{REC} \models \textit{Stein's Theorem}.$$

Thus, over $\mathsf{RCA}_0$ Stein Theorem has almost no strength, neither first order nor second order.

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# A Key Idea: How to use Whitehead property?

Let $F$ be a W-group with generators $\{x_1, x_2, \dots\}$. We build $G = \{y_0, y_1, y_2, \dots\}$ and $\pi : G \to F$ with $y_0 \mapsto 0_F$ and $y_i \mapsto x_i$.

If $\rho : F \to G$ splits $\pi$, them $\rho(x_i) = y_i + n_i y_0$ for some $n_i \in \mathbb{Z}$.

If we see a relation, say $\sigma := 3x_1 + x_2 = 0$ over $F$, we add a relation $3y_1 + y_2 + ky_0 = 0$ over $G$.

Since $\rho$ is a homomorphism, we must have $\rho(\sigma) = 0_G$, thus $3n_1 + n_2 = k$.

By playing $k = k(\sigma)$, we can diagonalize certain $\rho$ or code some information into $\rho$. Caution: $k(\sigma)$ must be a homomorphism.

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# Final Remark on Whitehead Problem

- Informal idea: Whitehead groups are free groups with bases outside the universe.
- In the reverse math setting, this is clearer: One could have a recursive group whose basis codes $0'$. (But it require $\mathrm{WKL}_0$ to turn it into W.)
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# Final Remarks on Reverse Mathematics

- Goal of Reverse Mathematics: What set existence axioms are needed to prove the theorems of ordinary, classical (countable) mathematics?
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