Computability Theory and Foundations of Mathematics
Tokyo, September 20-21, 2016

Abstract Booklet

Welcome to CTFM2016, the sixth conference in this series. Computability Theory and Foundations of Mathematics aims to develop computability theory and logical foundations of Mathematics. The scope involves the topics Computability Theory, Reverse Mathematics, Nonstandard Analysis, Proof Theory, Set Theory, Philosophy of Mathematics, Constructive Mathematics, Theory of Randomness and Computational Complexity Theory.

This conference is supported primarily by the Waseda Institute for Advanced Study and JSPS KAKENHI Grant Numbers JP26540001 and JP15H03634.

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Program

September 20

10:00 – 10:10 Opening

Invited session I

10:10 – 10:50 Kenshi Miyabe
Randomness notions in Muchnik and Medvedev degrees

11:10 – 11:50 Guohua Wu
When strong reduction is considered

(lunch break)

Invited session II

13:30 – 14:10 Wei Li
Ramsey's Theorem on Trees

14:20 – 15:00 Paul Shafer
Reverse mathematics and the strong Tietze extension theorem

(coffee break)

15:30 – 16:10 David Belanger
An Effective Perfect Set Theorem

Contributed session I

16:30 – 16:55 Helmut Schwichtenberg
Linear two-sorted constructive arithmetic

17:00 – 17:25 Emanuele Frittaion
Analyzing Size Change Termination in Reverse Mathematics

17:30 – 17:55 Keita Yokoyama
Trees with at most finitely many paths in reverse mathematics

(banquet 18:30 – 20:30)

September 21

Invited session III

9:30 – 10:10 Daisuke Ikegami
Gödel's Constructible Universe and logics

10:20 – 11:00 Philip Welch
Higher type recursion and Σ^0_3-Determinacy

11:10 – 11:50 Frank Stephan
Weakly Represented Families in Reverse Mathematics

(lunch break)

Invited session IV

13:30 – 14:10 Chi Tat Chong
Minimal degrees in weak subsystems of arithmetic

14:20 – 15:00 Yang Yue
A Lambda Calculus on Real Numbers

(coffee break)

Contributed session II

15:30 – 15:55 Takayuki Kihara
The Uniform Martin's Conjecture and the Wadge Degrees

16:00 – 16:25 Dávid Tóth
Towards Embedding Theorem: $Aut(\mathcal{D}_{\alpha e})$ embeds into $Aut(\mathcal{TOT}_{\alpha e})$

An Effective Perfect Set Theorem

David Belanger[1] and Keng Meng Ng[2]

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We gauge the difficulty of finding a perfect subtree in a tree of a given Cantor-Bendixson rank. To simplify the analysis we introduce half-derivative, and extend the definition of rank to include values of the form n-and-a-half; each increase of one-half in the rank corresponds to one added jump in the perfect-subtree problem.

Minimal degrees in weak subsystems of arithmetic

Chi Tat Chong

National University of Singapore

A set has minimal Turing degree if every set of strictly lower degree is recursive. Constructing a definable set of minimal degree without Σ_2 induction turns out to be a very challenging problem. In this talk we discuss the main difficulties encountered in such constructions and describe some properties of sets of minimal degree in models of Σ_2 bounding without Σ_2 induction.

Analyzing Size Change Termination in Reverse Mathematics

Emanuele Frittaion
(joint work with Silvia Steila, Keita Yokoyama, and Florian Pelupessy)

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Size Change Termination (SCT) is a general principle in termination analysis that has been widely studied in the last decade and successfully applied to all kinds of programs: functional programs (even high-order), imperative programs, logic programs. The idea behind SCT is not new: a program terminates if every infinite computation would cause an infinite descending sequence in some well-founded domain. However, already in the context of first-order functional languages, this principle is quite interesting. Note that in this context the well-founded domain is just the set of natural numbers.

In this work we consider SCT for first-order functional languages as formulated in [1,2,3].

Keywords: Reverse Mathematics, size-change-termination

References

1. Amir M. Ben-Amram. General size-change termination and lexicographic descent. In Torben Mogensen, David Schmidt, and I. Hal Sudborough, editors, *The Essence of Computation: Complexity, Analysis, Transformation. Essays Dedicated to Neil D. Jones*, volume 2566 of *Lecture Notes in Computer Science*, pages 3–17. Springer-Verlag, 2002.
2. Matthias Heizmann, Neil D. Jones, and Andreas Podelski. Size-change termination and transition invariants. In *SAS 2010, Perpignan, France, September 14-16, 2010. Proceedings*, pages 22–50, 2010.
3. Chin Soon Lee, Neil D. Jones, and Amir M. Ben-Amram. The size-change principle for program termination. In *Conference Record of POPL 2001: The 28th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, London, UK, January 17-19, 2001*, pages 81–92, 2001.

Gödel's Constructible Universe and logics

Daisuke Ikegami

Tokyo Denki University

In 1930s, Gödel proved that if ZF is consistent, then so is ZFC + GCH. For the proof, Gödel introduced the Constructible Universe L, that is the most basic model of set theory. In the construction of Gödel's L, one keeps adding 1st-order definable subsets of structures in question. If you replace "1st-order" with "2nd-order" in the construction, then you will get the model HOD, the class of hereditarily ordinal definable sets. While there are no measurable cardinals in L, the model HOD can accommodate most of large cardinals we have now, and it is one of the main themes in modern set theory to understand the connection between HOD and V (the class of all sets) under the existence of very large cardinals such as extendible cardinals. However, contrary to the case of Gödel's L, it is very difficult to analyze HOD, e.g., one can change the 1st-order theory of HOD easily by forcing. In this talk, we introduce inner models from Boolean valued higher logics and Woodin's Ω logic in the same way as Gödel's L from 1st-order logic, which are easier to analyze than HOD, and we discuss some basic properties of those models.

The Uniform Martin's Conjecture and the Wadge Degrees

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Assuming Woodin's AD^+, we show that, in a certain sense, the structure of "natural" many-one degrees is isomorphic to the Wadge degrees. Indeed, if $\mathcal{Q}$ is a better-quasi-order, the same holds true for $\mathcal{Q}$-valued many-one/Wadge degrees (e.g., many-one/Wadge degrees of k-partitions, k-coverings, ordinal-valued maps, etc.)

Formally, in this talk, a "natural" problem A is supposed to be *relativizable* and *uniformly degree invariant*, that is, if given two oracles X and Y are Turing equivalent, then the relativized problems A^X and A^Y are many-one equivalent, and moreover, one can effectively obtain a witness of $A^X \equiv_m A^Y$ from a witness of $X \equiv_T Y$. For instance, the halting problem (the Turing jump operation) and its (transfinite) iterations, the hyperjump operation, the sharp operation are all natural. Given uniformly degree invariant relativizable problems A, B, we say that A *is many-one-on-a-cone reducible to* B if there is an oracle C such that for any $X \geq_T C$, A^X is many-one reducible to B^X relative to C. We show that there is a natural one-to-one correspondence between the many-one-on-a-cone degrees of uniformly degree invariant relativizable decision problems ($\mathcal{Q}$-valued problems, resp.) and the Wadge degrees of subsets of Baire space ($\mathcal{Q}$-valued functions on Baire space, resp.)

This is joint work with Antonio Montalbán.

References

1. Takayuki Kihara and Antonio Montalbán, The uniform Martin's conjecture for many-one degrees, preprint.
2. Takayuki Kihara and Antonio Montalbán, On the structure of the Wadge degrees of BQO-valued Borel functions, in preparation.

* The author was partially supported by a Grant-in-Aid for JSPS fellows.

Ramsey's Theorem on Trees

Chi Tat Chong, Wei Li, Wei Wang, Yue Yang

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In this talk, we give a preliminary report on our study on Ramsey's Theorem on Trees in $B\Sigma_2$ models. Suppose M is a model of arithmetic. We use $2^{<M}$ to denote all finite binary strings in the sense of M. A coloring on $2^{<M}$ is a function $C : 2^{<M} \to k$, where $k \in M$. $T \subset 2^{<M}$ is called a *monochromatic tree* if $T \cong 2^{<M}$ and $C \upharpoonright T$ is a constant function. For every k, the principle that claims the existence of a monochromatic tree is called TT^1_k. TT^1 is $\forall k(TT^1_k)$.

Chubb, Hirst, and McNicholl [1] first introduced TT^1 and showed that $\mathrm{RCA}_0 + I\Sigma_2$ is sufficient to show TT^1 and TT^1 proves $B\Sigma_2$ over RCA_0. Corduan, Groszek and Mileti [2] proved that TT^1 is strictly stronger than $B\Sigma_2 +$ RCA_0. It is interesting to ask whether TT^1 is equivalent with $I\Sigma_2$. In this talk, we analyze several recursive colorings on the full binary trees in the model and their monochromatic subtrees. From there, we illustrate the logical strength of TT^1. This is a joint work with Chi Tat Chong, Wei Wang and Yue Yang.

References

1. Jennifer Chubb, Jeffry Hirst, Timothy McNicholl, *R*everse mathematics, computability, and partitions of trees, Journal of Symbolic Logic, Volume 74, Issue 1 (2009), 201 – 215.
2. Jared Corduan, Marcia J. Groszek, and Joseph R. Mileti, *Reverse mathematics and Ramsey's property for trees* Journal of Symbolic Logic, Volume 75, Issue 3 (2010), 945 – 954.

Randomness notions in Muchnik and Medvedev degrees

Kenshi Miyabe

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The main question of this talk is whether one can construct a more random set from a random set. This question can be formalized by mass problems. We view the elements of a class $P \subseteq 2^\omega$ as solutions to some problem. If a solution of a problem P can be constructed (or computable) from each solution of another problem Q, then we would say that P is more difficult than Q. We have two ways to formalize this notion. The difference between the two notions is uniformity.

Definition 1 (Muchnik [5], Medvedev [4]). *Let $P, Q \subseteq 2^\omega$.*

1. *We say that P is* Muchnik reducible (or weakly reducible) *to Q, denoted by $P \leq_w Q$, if, for every $f \in Q$, there is an element $g \leq_T f$ in P.*
2. *We say that P is* Medvedev reducible (or strongly reducible) *to Q, denoted by $P \leq_s Q$, if there is a Turing functional Φ such that $\Phi^f \in P$ for every $f \in Q$.*

Simpson [7] has already pointed out the importance of the class of ML-random sets in Muchnik degrees of Π^0_1 classes. We straightforwardly study some randomness notions in Muchnik and Medvedev degrees. The randomness notions we consider are ML-randomness, difference randomness, Demuth randomness, weakly 2-randomness, 2-randomness, computable randomness, Schnorr randomness, and Kurtz randomness. Each class is denoted by MLR, DiffR, DemR, W2R, 2R, CR, SR, and WR, respectively.

The obvious reductions $\leq_w$ and $\leq_s$ follows from inclusion of a randomness notion in another. Thus, our interest is only strictness, which can be interpreted as impossibility of (uniform) construction of a more random set from a random set.

For Muchnik degrees, we have the following result.

Theorem 1.

$$\mathrm{WR} <_w \mathrm{SR} \equiv_w \mathrm{CR} <_w \mathrm{MLR} \equiv_w \mathrm{DiffR} \begin{matrix} <_w \ \mathrm{W2R} \ <_w \\ <_w \ \mathrm{DemR} \ <_w \end{matrix} 2\mathrm{R}.$$

The proofs of the strictness are given by finding a degree with some property. For instance, CR $<_w$ MLR follows from the existence of a high minimal degree because each high degree contains a computably random set while each set Turing below a minimal degree cannot be ML-random by van Lambalgen's theorem.

The equivalence MLR $\equiv_w$ DiffR in Muchnik degrees is shown very interestingly. For every $A \oplus B \in$ MLR, at least one of A and B should be difference random because, if $A \geq_T \mathbf{0}'$, then B is 2-random, and thus difference random. However, we do not know which is difference random. In fact, we can not do this uniformly as the following result shows.

Theorem 2.

$$\mathrm{MLR} <_s \mathrm{DiffR}.$$

The proof uses the Levin-Kautz theorem on computable continuous measures [8, 3] and no-randomness-from-nothing for ML-randomness [2].

For the last part, we could separate SR and CR in Medvedev degrees.

Theorem 3.

$$\mathrm{SR} <_s \mathrm{CR}.$$

The proof uses the method to separate SR and CR [6], some results on randomness for computable measures [1], and Lévy's zero-one law from probability theory. This type of interaction between computability theory and analysis seems interesting to me.

Finally we remark that we can ask similar questions in reverse mathematics and Weihrauch degrees. One of the major differences between reverse mathematics and Weihrauch degrees is the uniformity. Thus, this work may help to clarify the difference.

This is a joint work with Rupert Hölzl.

References

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Linear two-sorted constructive arithmetic

Helmut Schwichtenberg (j.w.w. Stan Wainer, Leeds)

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We consider a term system in the style of Gödel's system T together with a computation model that allows to faithfully represent polynomial-time algorithms. To this end we follow a tradition initiated by Simmons (1988) and developed by Bellantoni and Cook (1992), where two sorts of variables are admitted, here called output and input variables. The idea is that input variables are the general ones which may be recursed on and used many times, whereas output variables cannot be recursed on and can be used only once. These ideas have been extended to a higher type setting and hence – via the Curry-Howard correspondence – to a linear two-sorted arithmetical system in the style of Heyting arithmetic in all finite types. Here we further extend this setup with the aim to also include certain nonlinear algorithms (like treesort), given by constants defined by equations involving multiple recursive calls.

Reverse mathematics and the strong Tietze extension theorem

Paul Shafer

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In second-order arithmetic, the *Tietze extension theorem* can be phrased by asserting that if X is a complete separable metric space, $C \subseteq X$ is closed, and $f\colon C \to \mathbb{R}$ is continuous and bounded, then there is a continuous and bounded extension F of f to all of X. This version of the Tietze extension theorem is provable in RCA_0 (see [2]). Giusto and Simpson introduced what they called the *strong Tietze extension theorem*, in which X is required to be compact and f and F are required to be uniformly continuous (in the sense of having moduli of uniform continuity) [1]. Giusto and Simpson showed that WKL_0 suffices to prove the strong Tietze extension theorem but that RCA_0 does not, which lead them to conjecture that the strong Tietze extension theorem is equivalent to WKL_0 over RCA_0. We confirm this conjecture.

References

1. Giusto, M., Simpson, S.G.: Located sets and reverse mathematics. Journal of Symbolic Logic 65(3), 1451–1480 (2000)
2. Simpson, S.G.: Subsystems of Second Order Arithmetic. Perspectives in Logic, Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, second edn. (2009)

Weakly Represented Families in Reverse Mathematics

Frank Stephan

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The talk gives an overview of the usage of weakly represented families in the following recent papers:

- Rupert Hoelzl, Sanjay Jain and Frank Stephan. *Inductive Inference and Reverse Mathematics.* Annals of Pure and Applied Logic, on Journal Homepage; was also at STACS 2015.
- Rupert Hoelzl, Dilip Raghavan, Frank Stephan and Jing Zhang. *Weakly Represented Families in Reverse Mathematics.* Rod Downey Festschrift, to appear in 2017.
- C.T. Chong, Rupert Hoelzl, Dilip Raghavan and Frank Stephan. *Maximal Almost Disjoint Families and Induction Axioms.* Paper under preparation.

Weakly represented families are families of sets recursive in one member A of the second-order part of the oracle such that some partial A-recursive function such that exactly the total members of this numbering coincide with the members of the family. These families allow to define axioms from recursion theory, inductive inference and cardinal invariants like DOM, MAD and MED in a natural way and allow to investigate these notions in the context of reverse mathematics. In particular it is shown that over $\mathrm{B}\Sigma_2$, MAD and DOM are complementary to each other. Similarly, over $\mathrm{I}\Sigma_2$, MED and AVOID are complementary to each other. DOM is in inductive inference also equivalent to Angluin's tell-tale criterion for the learnability of certain families of sets.

Towards Embedding Theorem: $Aut(\mathcal{D}_{\alpha e})$ embeds into $Aut(\mathcal{TOT}_{\alpha e})$

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Recently Cai, Ganchev, Lempp, Miller and Soskova proved that the total degrees $\mathcal{D}_T$ are definable in the enumeration degrees $\mathcal{D}_e$[1]: a degree is total iff it is trivial or a join of a maximal Kalimullin pair. The result may generalize to an α-level in higher computability theory for an admissible ordinal α. I will outline the proof of $Aut(\mathcal{D}_{\alpha e})$ embeds into $Aut(\mathcal{TOT}_{\alpha e})$ based on the 3 statements:

1. $\mathcal{TOT}_{\alpha e}$ degrees are embeddable in $\mathcal{D}_{\alpha e}$,
2. $\mathcal{TOT}_{\alpha e}$ are an automorphism base for $\mathcal{D}_{\alpha e}$,
3. $\mathcal{TOT}_{\alpha e}$ are definable in $\mathcal{D}_{\alpha e}$.

The discussion will centre across the main ideas and results.[1]

Keywords: α-computability theory, definability of the total degrees.

References

1. Mingzhong Cai, Hristo A Ganchev, Steffen Lempp, Joseph S. Miller, and Mariya I. Soskova. Defining totality in the enumeration degrees.

[1] This is an ongoing research in collaboration with Mariya Soskova.

Higher type recursion and Σ^0_3-Determinacy

Philip Welch

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We expand Kleene's 1960 treatment of Recursion in Higher Types in which he backed up his early equational calculus for recursion in finite types by giving it a Turing machine model. In type two the complete semi-recursive-in-J (the Turing jump) set of integers is recursively isomorphic to the complete G-Σ^0_1 (i.e. the complete Π^1_1) set of integers. We generalise this by using the Hamkins-Kidder model of infinite time turing machine and lift these results to Σ^0_3.

Theorem 1. *The complete G-Σ^0_3 set of integers is recursively isomorphic to the complete semi-recursive-in-eJ set of integers (where eJ is the ITTM eventual jump).*

We may then characterize the length of G-Σ^0_3-Monotone inductions.

When strong reduction is considered

Guohua Wu

Nanyang Technological University

In the talk, we first give a brief survey of c.e. wtt-degrees, the structure and the role of wtt-degrees in the study of Turing degrees. We will then introduce a concept of bounded-low degrees, proposed by Csima and her co-authors, and provide a couple of new results in this direction.

A Lambda Calculus on Real Numbers

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In the past CTFM meetings, I have talked about computation models on Baire spaces and real numbers. We identified a class of functions which can be characterized by both functional schemes and by master-slave machines. In this talk, I will show that this class of functions can also be characterized by λ-calculus with extra δ-reductions.

The talk is based on joint work with Duccio Pianigiani and Andrea Sorbi from University of Siena, Italy and Jiangjie Qiu from Renmin University, China.

Trees with at most finitely many paths in reverse mathematics

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It is well-known that any infinite computable tree $T \subseteq 2^{<\mathbb{N}}$ which has at most finitely many paths has a computable path. Then, how can we understand this in the context of reverse mathematics? Is it provable within RCA_0? Actually, we need to formalize the statement carefully. Within RCA_0, the assertion "every infinite binary tree which has at most one path has a path" is already equivalent to WKL since the negation of WKL implies the existence of an infinite binary tree with no path. In this talk, we will consider the following very weak versions of König's lemma.

Definition 1. 1. WKL(pf-*bd*)*: an infinite binary tree* $T \subseteq 2^{<\mathbb{N}}$ *has a path if there exists* $c \in \mathbb{N}$ *such that for any prefix-free set* $P \subseteq T$, $|P| \leq c$.

2. WKL(w-*bd*)*: an infinite binary tree* $T \subseteq 2^{<\mathbb{N}}$ *has a path if there exists* $c \in \mathbb{N}$ *such that for any* $n \in \mathbb{N}$, $|T^{=n}| \leq c$, *where* $T^{=n} = \{\sigma \in T \mid \mathrm{lh}(\sigma) = n\}$.
3. WKL(ext-*bd*)*: an infinite binary tree* $T \subseteq 2^{<\mathbb{N}}$ *has a path if there exists* $c \in \mathbb{N}$ *such that for any* $n \in \mathbb{N}$, $|T^{=n}_{\mathrm{ext}}| \leq c$, *where* $T^{=n}_{\mathrm{ext}} = \{\sigma \in T \mid \mathrm{lh}(\sigma) = n \wedge \sigma \text{ is extendible}\}$.
4. KL(pf-*bd*)*: an infinite binary tree* $T \subseteq \mathbb{N}^{<\mathbb{N}}$ *has a path if there exists* $c \in \mathbb{N}$ *such that for any prefix-free set* $P \subseteq T$, $|P| \leq c$.
5. KL(w-*bd*)*: an infinite binary tree* $T \subseteq \mathbb{N}^{<\mathbb{N}}$ *has a path if there exists* $c \in \mathbb{N}$ *such that for any* $n \in \mathbb{N}$, $|T^{=n}| \leq c$.
6. KL(ext-*bd*)*: an finitely-branching infinite binary tree* $T \subseteq \mathbb{N}^{<\mathbb{N}}$ *has a path if there exists* $c \in \mathbb{N}$ *such that for any* $n \in \mathbb{N}$, $|T^{=n}_{\mathrm{ext}}| \leq c$.

(Here, $T^{=n}$ *denotes the set of all strings of length* n *in* T*, and* T_{ext} *denotes the set of all extendible strings in* T*.)*

In fact, 1,2,3,4,5 are all true in any ω-model of RCA_0, while 4 and 5 actually need some non-trivial induction. On the other hand, 1,2,3 are provable from WKL_0, so they won't imply any extra induction. Still, 2 and 3 require some induction "conditionally". We will see this more precisely. This is a joint work with Stephen G. Simpson.

* This work is partially supported by JSPS KAKENHI grant number 16K17640, JSPS fellowship for research abroad.