Minimal Degrees in Models of Weak Subsystems of Arithmetic

C T Chong

National University of Singapore

chongct@math.nus.edu.sg

21 September 2016

Minimal Degree

Definition

X has minimal degree if every $Y <_T X$ is recursive.

- Spector (1957) first constructed a set of minimal degree.
- Sacks (1961) constructed a minimal degree $< \mathbf{0}'$.
- Typical construction of a set of minimal degree applies the "tree method".

Minimal Degree

Definition

X has minimal degree if every $Y <_T X$ is recursive.

- Spector (1957) first constructed a set of minimal degree.
- Sacks (1961) constructed a minimal degree $< \mathbf{0}'$.
- Typical construction of a set of minimal degree applies the "tree method".

Minimal Degree

Definition

X has minimal degree if every $Y <_T X$ is recursive.

- Spector (1957) first constructed a set of minimal degree.
- Sacks (1961) constructed a minimal degree $< \mathbf{0}'$.
- Typical construction of a set of minimal degree applies the "tree method".

Minimal Degree

Definition

X has minimal degree if every $Y <_T X$ is recursive.

- Spector (1957) first constructed a set of minimal degree.
- Sacks (1961) constructed a minimal degree $< \mathbf{0}'$.
- Typical construction of a set of minimal degree applies the "tree method".

Minimal Degree

Definition

X has minimal degree if every $Y <_T X$ is recursive.

- Spector (1957) first constructed a set of minimal degree.
- Sacks (1961) constructed a minimal degree $< \mathbf{0}'$.
- Typical construction of a set of minimal degree applies the “tree method”.

The Tree Method

Given Φ_e and an infinite recursive tree $T \subset 2^{<\omega}$, define by recursion the *splitting subree* $Sp(e, T) \subset T$ such that

- If $\tau_1, \tau_2 \in Sp(e, T)$ are incomparable, then $\Phi_e^{\tau_1}(x) \neq \Phi_e^{\tau_2}(x)$ for some x.

There are two possibilities:

1. (Splitting tree) Every $\tau \in Sp(e, T)$ has a (least) pair of incomparable extensions. in $Sp(e, T)$. Let $T_e = Sp(e, T)$. Then if $X \in [T_e]$, $\Phi_e^X \equiv_T X$;
2. (Full tree) There is a $\tau \in Sp(e, T)$ with no extension in $Sp(e, T)$ (i.e. a dead end). Let $T_e = \{\tau' \in T : \tau' \succeq \tau\}$. Then any $X \in [T_e]$ satisfies Φ_e^X is partial or Φ_e^X is recursive.

The Tree Method

Given Φ_e and an infinite recursive tree $T \subset 2^{<\omega}$, define by recursion the *splitting subree* $Sp(e, T) \subset T$ such that

- If $\tau_1, \tau_2 \in Sp(e, T)$ are incomparable, then $\Phi_e^{\tau_1}(x) \neq \Phi_e^{\tau_2}(x)$ for some x.

There are two possibilities:

1. (Splitting tree) Every $\tau \in Sp(e, T)$ has a (least) pair of incomparable extensions. in $Sp(e, T)$. Let $T_e = Sp(e, T)$. Then if $X \in [T_e]$, $\Phi_e^X \equiv_T X$;
2. (Full tree) There is a $\tau \in Sp(e, T)$ with no extension in $Sp(e, T)$ (i.e. a dead end). Let $T_e = \{\tau' \in T : \tau' \succeq \tau\}$. Then any $X \in [T_e]$ satisfies Φ_e^X is partial or Φ_e^X is recursive.

The Tree Method

Given Φ_e and an infinite recursive tree $T \subset 2^{<\omega}$, define by recursion the *splitting subree* $Sp(e, T) \subset T$ such that

- If $\tau_1, \tau_2 \in Sp(e, T)$ are incomparable, then $\Phi_e^{\tau_1}(x) \neq \Phi_e^{\tau_2}(x)$ for some x.

There are two possibilities:

1. (Splitting tree) Every $\tau \in Sp(e, T)$ has a (least) pair of incomparable extensions. in $Sp(e, T)$. Let $T_e = Sp(e, T)$. Then if $X \in [T_e]$, $\Phi_e^X \equiv_T X$;
2. (Full tree) There is a $\tau \in Sp(e, T)$ with no extension in $Sp(e, T)$ (i.e. a dead end). Let $T_e = \{\tau' \in T : \tau' \succeq \tau\}$. Then any $X \in [T_e]$ satisfies Φ_e^X is partial or Φ_e^X is recursive.

The Tree Method

Given Φ_e and an infinite recursive tree $T \subset 2^{<\omega}$, define by recursion the *splitting subree* $Sp(e, T) \subset T$ such that

- If $\tau_1, \tau_2 \in Sp(e, T)$ are incomparable, then $\Phi_e^{\tau_1}(x) \neq \Phi_e^{\tau_2}(x)$ for some x.

There are two possibilities:

1. (Splitting tree) Every $\tau \in Sp(e, T)$ has a (least) pair of incomparable extensions. in $Sp(e, T)$. Let $T_e = Sp(e, T)$. Then if $X \in [T_e]$, $\Phi_e^X \equiv_T X$;
2. (Full tree) There is a $\tau \in Sp(e, T)$ with no extension in $Sp(e, T)$ (i.e. a dead end). Let $T_e = \{\tau' \in T : \tau' \succeq \tau\}$. Then any $X \in [T_e]$ satisfies Φ_e^X is partial or Φ_e^X is recursive.

The Tree Method

- Starting with $T = 2^{<\omega}$, define $T_0 \supset T_1 \supset \cdots$.
- Any $X \in \bigcap_e [T_e]$ has minimal degree. There is an $X <_T \emptyset''$.
- The split into (1) or (2) is a $\emptyset''$-decision.
- Σ^0_2 induction is sufficient to implement the Spector/Sacks construction.

Question. Is there a set of minimal degree in the absence of Σ^0_2 induction?

The Tree Method

- Starting with $T = 2^{<\omega}$, define $T_0 \supset T_1 \supset \cdots$.
- Any $X \in \bigcap_e [T_e]$ has minimal degree. There is an $X <_T \emptyset''$.
- The split into (1) or (2) is a $\emptyset''$-decision.
- Σ^0_2 induction is sufficient to implement the Spector/Sacks construction.

Question. Is there a set of minimal degree in the absence of Σ^0_2 induction?

The Tree Method

- Starting with $T = 2^{<\omega}$, define $T_0 \supset T_1 \supset \cdots$.
- Any $X \in \bigcap_e [T_e]$ has minimal degree. There is an $X <_T \emptyset''$.
- The split into (1) or (2) is a $\emptyset''$-decision.
- Σ^0_2 induction is sufficient to implement the Spector/Sacks construction.

Question. Is there a set of minimal degree in the absence of Σ^0_2 induction?

The Tree Method

- Starting with $T = 2^{<\omega}$, define $T_0 \supset T_1 \supset \cdots$.
- Any $X \in \bigcap_e [T_e]$ has minimal degree. There is an $X <_T \emptyset''$.
- The split into (1) or (2) is a $\emptyset''$-decision.
- Σ^0_2 induction is sufficient to implement the Spector/Sacks construction.

Question. Is there a set of minimal degree in the absence of Σ^0_2 induction?

The Tree Method

- Starting with $T = 2^{<\omega}$, define $T_0 \supset T_1 \supset \cdots$.
- Any $X \in \bigcap_e [T_e]$ has minimal degree. There is an $X <_T \emptyset''$.
- The split into (1) or (2) is a $\emptyset''$-decision.
- Σ^0_2 induction is sufficient to implement the Spector/Sacks construction.

Question. Is there a set of minimal degree in the absence of Σ^0_2 induction?

Models of $P^- + I\Sigma^0_1 + \neg I\Sigma^0_2$

Fix $\mathfrak{M} = (M, +, \cdot, 0, 1) \models P^- + I\Sigma^0_1 + \neg I\Sigma^0_2$. Let I be a Σ^0_2 cut with a Σ^0_2 cofinal $g : I \to M$.

- (Tame cut) If $\mathfrak{M} \models B\Sigma^0_2$, then g may be chosen to be strictly increasing with a recursive approximation g', i.e. $g(i) = \lim_s g'(s, i)$ for $i \in I$.
- (Bitame cut [Chong, Lempp and Yang (2010)]) If $\mathfrak{M} \models \neg B\Sigma^0_2$, then I, g may be chosen so that g is cofinal, strictly increasing on I, and (reverse) cofinal, strictly decreasing on $a \setminus I$ for some $a > I$.

Models of $P^- + I\Sigma^0_1 + \neg I\Sigma^0_2$

Fix $\mathfrak{M} = (M, +, \cdot, 0, 1) \models P^- + I\Sigma^0_1 + \neg I\Sigma^0_2$. Let I be a Σ^0_2 cut with a Σ^0_2 cofinal $g : I \to M$.

- (Tame cut) If $\mathfrak{M} \models B\Sigma^0_2$, then g may be chosen to be strictly increasing with a recursive approximation g', i.e. $g(i) = \lim_s g'(s, i)$ for $i \in I$.
- (Bitame cut [Chong, Lempp and Yang (2010)]) If $\mathfrak{M} \models \neg B\Sigma^0_2$, then I, g may be chosen so that g is cofinal, strictly increasing on I, and (reverse) cofinal, strictly decreasing on $a \setminus I$ for some $a > I$.

Models of $P^- + I\Sigma_1^0 + \neg I\Sigma_2^0$

Fix $\mathfrak{M} = (M, +, \cdot, 0, 1) \models P^- + I\Sigma_1^0 + \neg I\Sigma_2^0$. Let I be a Σ_2^0 cut with a Σ_2^0 cofinal $g : I \to M$.

- (Tame cut) If $\mathfrak{M} \models B\Sigma_2^0$, then g may be chosen to be strictly increasing with a recursive approximation g', i.e. $g(i) = \lim_s g'(s, i)$ for $i \in I$.
- (Bitame cut [Chong, Lempp and Yang (2010)]) If $\mathfrak{M} \models \neg B\Sigma_2^0$, then I, g may be chosen so that g is cofinal, strictly increasing on I, and (reverse) cofinal, strictly decreasing on $a \setminus I$ for some $a > I$.

Tree Method in $\neg I\Sigma_2^0$

Fix I, g, a as above.

- For $i \leq a$, let $\Phi_i^\sigma(x) = \sigma(x)$ if $x \leq |\sigma|$ and $g'(x, i) \neq g'(x+1, i)$, and $\Phi_i^\sigma(x) = 0$ otherwise.
- Then for $i \in I$, T_i is a full tree with root of length $\geq g(i)$.
- Let $T = 2^{<M}$ and define $Sp(i, T_i$ as before. Then T_i is *not* defined for $i \notin I$.

Hence the Spector/Sacks construction fails.

Question. Is there a set of minimal degree $<_T \emptyset'$ or $<_T \emptyset''$ in $\mathfrak{M}$?

Tree Method in $\neg I\Sigma_2^0$

Fix I, g, a as above.

- For $i \leq a$, let $\Phi_i^\sigma(x) = \sigma(x)$ if $x \leq |\sigma|$ and $g'(x, i) \neq g'(x+1, i)$, and $\Phi_i^\sigma(x) = 0$ otherwise.
- Then for $i \in I$, T_i is a full tree with root of length $\geq g(i)$.
- Let $T = 2^{<M}$ and define $Sp(i, T_i$ as before. Then T_i is *not* defined for $i \notin I$.

Hence the Spector/Sacks construction fails.

Question. Is there a set of minimal degree $<_T \emptyset'$ or $<_T \emptyset''$ in $\mathfrak{M}$?

Tree Method in $\neg I\Sigma_2^0$

Fix I, g, a as above.

- For $i \leq a$, let $\Phi_i^\sigma(x) = \sigma(x)$ if $x \leq |\sigma|$ and $g'(x, i) \neq g'(x+1, i)$, and $\Phi_i^\sigma(x) = 0$ otherwise.
- Then for $i \in I$, T_i is a full tree with root of length $\geq g(i)$.
- Let $T = 2^{<M}$ and define $Sp(i, T_i$ as before. Then T_i is *not* defined for $i \notin I$.

Hence the Spector/Sacks construction fails.

Question. Is there a set of minimal degree $<_T \emptyset'$ or $<_T \emptyset''$ in $\mathfrak{M}$?

Tree Method in $\neg I\Sigma_2^0$

Fix I, g, a as above.

- For $i \leq a$, let $\Phi_i^\sigma(x) = \sigma(x)$ if $x \leq |\sigma|$ and $g'(x,i) \neq g'(x+1,i)$, and $\Phi_i^\sigma(x) = 0$ otherwise.
- Then for $i \in I$, T_i is a full tree with root of length $\geq g(i)$.
- Let $T = 2^{<M}$ and define $Sp(i, T_i$ as before. Then T_i is *not* defined for $i \notin I$.

Hence the Spector/Sacks construction fails.

Question. Is there a set of minimal degree $<_T \emptyset'$ or $<_T \emptyset''$ in $\mathfrak{M}$?

Tree Method in $\neg I\Sigma_2^0$

Fix I, g, a as above.

- For $i \leq a$, let $\Phi_i^\sigma(x) = \sigma(x)$ if $x \leq |\sigma|$ and $g'(x, i) \neq g'(x+1, i)$, and $\Phi_i^\sigma(x) = 0$ otherwise.
- Then for $i \in I$, T_i is a full tree with root of length $\geq g(i)$.
- Let $T = 2^{<M}$ and define $Sp(i, T_i$ as before. Then T_i is *not* defined for $i \notin I$.

Hence the Spector/Sacks construction fails.

Question. Is there a set of minimal degree $<_T \emptyset'$ or $<_T \emptyset''$ in $\mathfrak{M}$?

Minimal Degrees in $\neg I\Sigma_2^0$

$X \subset M$ is *regular* if $X \restriction s$ is $\mathfrak{M}$-finite for every $s \in M$.

- (Chong and Mourad 1990) There is an $\mathfrak{M} \models B\Sigma_2^0$ in which $\omega = I$ is a set of minimal degree.
- $I <_T \emptyset''$ and nonregular.
- If $\mathfrak{M} \models I\Sigma_1^0$ is countable, then there is a regular set X of minimal degree. But X may not be definable.

Refined Question. Is there a *regular* set of minimal degree $<_T \emptyset''$ or $<_T \emptyset'$?

Minimal Degrees in $\neg I\Sigma_2^0$

$X \subset M$ is *regular* if $X \upharpoonright s$ is $\mathfrak{M}$-finite for every $s \in M$.

- (Chong and Mourad 1990) There is an $\mathfrak{M} \models B\Sigma_2^0$ in which $\omega = I$ is a set of minimal degree.
- $I <_T \emptyset''$ and nonregular.
- If $\mathfrak{M} \models I\Sigma_1^0$ is countable, then there is a regular set X of minimal degree. But X may not be definable.

Refined Question. Is there a *regular* set of minimal degree $<_T \emptyset''$ or $<_T \emptyset'$?

Minimal Degrees in $\neg I\Sigma^0_2$

$X \subset M$ is *regular* if $X \upharpoonright s$ is $\mathfrak{M}$-finite for every $s \in M$.

- (Chong and Mourad 1990) There is an $\mathfrak{M} \models B\Sigma^0_2$ in which $\omega = I$ is a set of minimal degree.
- $I <_T \emptyset''$ and nonregular.
- If $\mathfrak{M} \models I\Sigma^0_1$ is countable, then there is a regular set X of minimal degree. But X may not be definable.

Refined Question. Is there a *regular* set of minimal degree $<_T \emptyset''$ or $<_T \emptyset'$?

Minimal Degrees in $\neg I\Sigma_2^0$

$X \subset M$ is *regular* if $X \restriction s$ is $\mathfrak{M}$-finite for every $s \in M$.

- (Chong and Mourad 1990) There is an $\mathfrak{M} \models B\Sigma_2^0$ in which $\omega = I$ is a set of minimal degree.
- $I <_T \emptyset''$ and nonregular.
- If $\mathfrak{M} \models I\Sigma_1^0$ is countable, then there is a regular set X of minimal degree. But X may not be definable.

Refined Question. Is there a *regular* set of minimal degree $<_T \emptyset''$ or $<_T \emptyset'$?

Minimal Degrees in $\neg I\Sigma_2^0$

$X \subset M$ is *regular* if $X \upharpoonright s$ is $\mathfrak{M}$-finite for every $s \in M$.

- (Chong and Mourad 1990) There is an $\mathfrak{M} \models B\Sigma_2^0$ in which $\omega = I$ is a set of minimal degree.
- $I <_T \emptyset''$ and nonregular.
- If $\mathfrak{M} \models I\Sigma_1^0$ is countable, then there is a regular set X of minimal degree. But X may not be definable.

Refined Question. Is there a *regular* set of minimal degree $<_T \emptyset''$ or $<_T \emptyset'$?

Minimal Degrees in $\neg I\Sigma_2$

Theorem

Let $\mathfrak{M} \models P^- + B\Sigma^0_2 + \neg I\Sigma^0_2$. Every regular set of minimal degree $<_T \emptyset''$ in $\mathfrak{M}$ is low, i.e. if $X <_T \emptyset''$ has minimal degree, then $X <_T \emptyset'$ and $X' \equiv_T \emptyset'$.

Theorem

There is a model $\mathfrak{M}$ of $P^- + I\Sigma^0_1 + \neg B\Sigma^0_2$ with a set X of minimal degree $<_T \emptyset'$ preserving $I\Sigma^0_1$, i.e, $\mathfrak{M}[X] \models I\Sigma^0_1$. In particular,

$$RCA_0 + \text{“There is a minimal degree”}$$

does not imply $B\Sigma^0_2$.

Minimal Degrees in $\neg I\Sigma_2$

Theorem

Let $\mathfrak{M} \models P^- + B\Sigma^0_2 + \neg I\Sigma^0_2$. Every regular set of minimal degree $<_T \emptyset''$ in $\mathfrak{M}$ is low, i.e. if $X <_T \emptyset''$ has minimal degree, then $X <_T \emptyset'$ and $X' \equiv_T \emptyset'$.

Theorem

There is a model $\mathfrak{M}$ of $P^- + I\Sigma^0_1 + \neg B\Sigma^0_2$ with a set X of minimal degree $<_T \emptyset'$ preserving $I\Sigma^0_1$, i.e, $\mathfrak{M}[X] \models I\Sigma^0_1$. In particular,

$$RCA_0 + \text{“There is a minimal degree”}$$

does not imply $B\Sigma^0_2$.

Minimal Degrees in $\neg I\Sigma_2$

Theorem

Let $\mathfrak{M} \models P^- + B\Sigma^0_2 + \neg I\Sigma^0_2$. Every regular set of minimal degree $<_T \emptyset''$ in $\mathfrak{M}$ is low, i.e. if $X <_T \emptyset''$ has minimal degree, then $X <_T \emptyset'$ and $X' \equiv_T \emptyset'$.

Theorem

There is a model $\mathfrak{M}$ of $P^- + I\Sigma^0_1 + \neg B\Sigma^0_2$ with a set X of minimal degree $<_T \emptyset'$ preserving $I\Sigma^0_1$, i.e, $\mathfrak{M}[X] \models I\Sigma^0_1$. In particular,

$$RCA_0 + \text{“There is a minimal degree”}$$

does not imply $B\Sigma^0_2$.

Minimal Degree in $\neg I\Sigma_2^0$

Question. Is there a model of

$$\mathrm{RCA}_0 + B\Sigma_2^0 + \neg I\Sigma_2^0 + \text{“There is a minimal degree”?}$$

More generally,

Question. Given a finite partial ordering $\mathbb{P}$, is there a model of $\mathrm{RCA}_0 + B\Sigma_2^0 + \neg I\Sigma_2^0$ whose second order elements are isomorphic to $\mathbb{P}$ under Turing reducibility?

Comparison With α-recursion:

- It is not known if there is a minimal $\aleph_\omega^L$-degree.
- It is not known if, for X minimal, $X <_\alpha \emptyset''$ implies $X <_\alpha \emptyset'$, where $\alpha = \aleph_\omega^L$, although in this case every set below $\emptyset'$ is low.

Minimal Degree in $\neg I\Sigma_2^0$

Question. Is there a model of

$$\mathrm{RCA}_0 + B\Sigma_2^0 + \neg I\Sigma_2^0 + \text{“There is a minimal degree”?}$$

More generally,

Question. Given a finite partial ordering $\mathbb{P}$, is there a model of $\mathrm{RCA}_0 + B\Sigma_2^0 + \neg I\Sigma_2^0$ whose second order elements are isomorphic to $\mathbb{P}$ under Turing reducibility?

Comparison With α-recursion:

- It is not known if there is a minimal $\aleph_\omega^L$-degree.
- It is not known if, for X minimal, $X <_\alpha \emptyset''$ implies $X <_\alpha \emptyset'$, where $\alpha = \aleph_\omega^L$, although in this case every set below $\emptyset'$ is low.

Minimal Degree in $\neg I\Sigma_2^0$

Question. Is there a model of

$$\mathrm{RCA}_0 + B\Sigma_2^0 + \neg I\Sigma_2^0 + \text{“There is a minimal degree”?}$$

More generally,

Question. Given a finite partial ordering $\mathbb{P}$, is there a model of $\mathrm{RCA}_0 + B\Sigma_2^0 + \neg I\Sigma_2^0$ whose second order elements are isomorphic to $\mathbb{P}$ under Turing reducibility?

Comparison With α-recursion:

- It is not known if there is a minimal $\aleph_\omega^L$-degree.
- It is not known if, for X minimal, $X <_\alpha \emptyset''$ implies $X <_\alpha \emptyset'$, where $\alpha = \aleph_\omega^L$, although in this case every set below $\emptyset'$ is low.

Minimal Degree in $\neg I\Sigma_2^0$

Question. Is there a model of

$$\mathrm{RCA}_0 + B\Sigma_2^0 + \neg I\Sigma_2^0 + \text{“There is a minimal degree”?}$$

More generally,

Question. Given a finite partial ordering $\mathbb{P}$, is there a model of $\mathrm{RCA}_0 + B\Sigma_2^0 + \neg I\Sigma_2^0$ whose second order elements are isomorphic to $\mathbb{P}$ under Turing reducibility?

Comparison With α-recursion:

- It is not known if there is a minimal $\aleph_\omega^L$-degree.
- It is not known if, for X minimal, $X <_\alpha \emptyset''$ implies $X <_\alpha \emptyset'$, where $\alpha = \aleph_\omega^L$, although in this case every set below $\emptyset'$ is low.

Minimal Degree in $\neg I\Sigma_2^0$

Question. Is there a model of

$$\mathrm{RCA}_0 + B\Sigma_2^0 + \neg I\Sigma_2^0 + \text{“There is a minimal degree”?}$$

More generally,

Question. Given a finite partial ordering $\mathbb{P}$, is there a model of $\mathrm{RCA}_0 + B\Sigma_2^0 + \neg I\Sigma_2^0$ whose second order elements are isomorphic to $\mathbb{P}$ under Turing reducibility?

Comparison With α-recursion:

- It is not known if there is a minimal $\aleph_\omega^L$-degree.
- It is not known if, for X minimal, $X <_\alpha \emptyset''$ implies $X <_\alpha \emptyset'$, where $\alpha = \aleph_\omega^L$, although in this case every set below $\emptyset'$ is low.