Inner models from Boolean valued higher-order logics and Ω-logic

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From now on...

We work in ZFC unless clearly specified.

V is the class of all sets.

Gödel's Constructible Hierarchy

Theorem (Gödel)

If ZF is consistent, then so is ZFC + GCH.

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Definition

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\begin{aligned}
\mathrm{L}_0 &= \emptyset, \\
\mathrm{L}_{\alpha+1} &= \mathsf{Def}_{\mathsf{FOL}}\big((\mathrm{L}_\alpha, \in)\big), \\
\mathrm{L}_\gamma &= \bigcup_{\alpha<\gamma} \mathrm{L}_\alpha \qquad (\gamma \text{ is limit}), \\
\mathrm{L} &= \bigcup_{\alpha \in \mathsf{On}} \mathrm{L}_\alpha .
\end{aligned}
$$

Inner models from logics

Definition

Given a logic $\mathcal{L}$ with a definability notion,

$$\mathrm{L}_0(\mathcal{L}) = \emptyset,$$

$$\mathrm{L}_{\alpha+1}(\mathcal{L}) = \mathsf{Def}_{\mathcal{L}}\big((\mathrm{L}_\alpha(\mathcal{L}), \in)\big),$$

$$\mathrm{L}_\gamma(\mathcal{L}) = \bigcup_{\alpha<\gamma} \mathrm{L}_\alpha(\mathcal{L}) \quad (\gamma \text{ is limit}),$$

$$\mathrm{L}(\mathcal{L}) = \bigcup_{\alpha\in\mathsf{On}} \mathrm{L}_\alpha(\mathcal{L}).$$

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Question

What is $\mathrm{L}(\mathcal{L})$ if $\mathcal{L}$ is full 2nd-order logic (SOL)?

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HOD!

What is HOD?

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- x is OD if x is 1st-order definable in the structure $(V, \in)$ with an ordinal parameter.
- $x \in$ HOD if every element of tr.cl.$(\{x\})$ is OD.
  Note: tr.cl.$(\{x\})$ is the least transitive set y such that $x \in y$.

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- HOD is a transitive model of ZFC.
- HOD is the largest transitive proper class s.t. every set in the model is OD.
- HOD can accommodate all the large cardinals we have so far.
- HOD is very "non-absolute" (e.g., for any real x, one can force "$x \in$ HOD" in a set forcing extension).
- One cannot decide e.g., whether HOD satisfies CH.

Inner models from higher order logics

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Answer

It is the same as HOD.

Inner models from logics ctd.

Question

What about other logics?

Kennedy, Magidor, and Väänänen explored on inner models from first order logic with "generalized quantifiers".

Inner models from logics ctd.

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In this talk, we will discuss inner models from Boolean valued higher order logics and Woodin's Ω-logic.

Inner models from logics: Goal & Motivation

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Construct a model of set theory which is "close to" HOD but easier to analyze.

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Theorem (Woodin)

Let κ be extendible. Then exactly one of the following holds:

1. for every regular $\gamma > \kappa$, γ is inaccessible in HOD, OR
2. for every singular cardinal $\gamma > \kappa$, γ is singular in HOD and $(\gamma^+)^{\mathsf{HOD}} = \gamma^+$.

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Definition (Woodin)

HOD Conjecture states that the latter case in the above theorem holds.

Inner models from logics: Motivation ctd.

1. HOD Conjecture is connected to the Inner Model Program for a supercompact cardinal.
2. HOD Conjecture has an application to the problem on the existence of non-trivial elementary embeddings from V to itself in ZF.

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To solve HOD Conjecture, one would expect a fine analysis of HOD. But HOD is very "non-absolute", e.g.,

Proposition (Folklore)

For any real x, there is a partial order P such that "$x \in \mathrm{HOD}$" in V^P.

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Proposition (Folklore)

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Question

Can one construct a model of set theory which is "close to" HOD, but invariant under forcing extensions?

Boolean valued 2nd-order logic: background

Two semantics for 2nd-order logic:

1. Full semantics: Highly complex (very powerful), does not enjoy completeness, ω-compactness.
2. Henkin semantics: Very simple (very week), enjoys completeness, ω-compactness.

Boolean valued 2nd-order logic: background

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2. Henkin semantics: Very simple (very week), enjoys completeness, ω-compactness.

Boolean valued second order logic is a powerful logic sitting between the two semantics.

Boolean valued 2nd-order logic: Boolean valued structures

Definition

Let $\mathcal{L}$ be a relational language. A Boolean valued $\mathcal{L}$-structure is a tuple $M = (A, \mathbb{B}, \{R_i^M\})$ where

1. A is a nonempty set,
2. $\mathbb{B}$ is a complete Boolean algebra, and
3. for each n-ary relational symbol R_i in $\mathcal{L}$, $R_i^M \colon A^n \to \mathbb{B}$.

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2. $\mathbb{B}$ is a complete Boolean algebra, and
3. for each n-ary relational symbol R_i in $\mathcal{L}$, $R_i^M : A^n \to \mathbb{B}$.

Example

If $\mathbb{B} = \{0, 1\}$, each R_i^M is a relation in 1st-order logic and M is the same as a 1st-order structure.

Truth of 2nd-order formulas in Boolean valued structures

Basic idea: "subsets" are functions from A to $\mathbb{B}$.

Definition

Let $M = (A, \mathbb{B}, \{R_i\})$ be a Boolean valued $\mathcal{L}$-structure. Then we assign $\|\phi[\vec{a}, \vec{f}]\|^M \in \mathbb{B}$ to each 2nd-order formula ϕ, $\vec{a} \in {}^{<\omega}A$, and $\vec{f} \in {}^{<\omega}({}^{A}\mathbb{B})$ as follows:

1. ϕ is $R_i(\vec{x})$. Then $\|R_i(\vec{x})[\vec{a}]\|^M = R_i^M(\vec{a})$.
2. ϕ is $X(x)$. Then $\|X(x)[a, f]\|^M = f(a)$.
3. Boolean combinations are as usual.
4. ϕ is $\exists x \psi$. Then $\|\exists x \psi[\vec{a}, \vec{f}]\|^M = \bigvee_{b \in A} \|\psi[b, \vec{a}, \vec{f}]\|^M$.
5. ϕ is $\exists X \psi$. Then $\|\exists X \psi[\vec{a}, \vec{f}]\|^M = \bigvee_{g:\, A \to \mathbb{B}} \|\psi[\vec{a}, g, \vec{f}]\|^M$.

Boolean valued 2nd-order logic: Boolean-validity

Definition

Let $\mathcal{L}$ be relational. A 2nd-order $\mathcal{L}$-sentence ϕ is Boolean-valid if $\|\phi\|^M = 1$ for any Boolean valued $\mathcal{L}$-structure M.

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Lemma

A 2nd-order $\mathcal{L}$-sentence ϕ is Boolean-valid if and only if for any *1st-order* $\mathcal{L}$-structure M, a partial order $\mathbb{P}$, and a $\mathbb{P}$-generic filter G over V, $(M, \mathcal{P}(M)^{V[G]}) \models \phi$.

Boolean valued 2nd-order logic: Definability

Definition

1. For a set A, $\vec{a} \in A^{<\omega}$, and a second order formula ϕ, $(\phi, \vec{a})$ is suitable to A if for every element x of A, either $\phi[x, \vec{a}]$ or $\neg\phi[x, \vec{a}]$ is Boolean valid with the first order universe A.

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2. Let $(\phi, \vec{a})$ be suitable to A. Then a set $X \subseteq A$ is BVSOL-definable via $(\phi, \vec{a})$ if X is the collection of $x \in A$ such that $\phi[x, \vec{a}]$ is Boolean valid with the first order universe A.

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3. $\mathrm{Def}_{2b}(A)$ is the collection of BVSOL-definable subsets of A via some $(\phi, \vec{a})$ suitable to A.

Boolean valued 2nd-order logic: Definability

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3. $\mathrm{Def}_{2b}(A)$ is the collection of BVSOL-definable subsets of A via some $(\phi, \vec{a})$ suitable to A.

One can introduce the constructible hierarchy & universe w.r.t. BVSOL. We write L_{α}^{2b} and L^{2b} for those.

Inner models from logics: L^{2b}

Remark

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Is L^{2b} a model of AC under the existence of large cardinals?

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Remark

If $V = \mathrm{L}$, then $\mathrm{L} = \mathrm{L}^{2b} = \mathrm{HOD}$.

Inner models from logics: L^{2b} ctd.

Theorem

Assuming large cardinals, one can show that

$$(\omega, \mathcal{P}(\omega), \in, 0, 1, +, \cdot)^{\mathrm{L}^{2b}} \prec (\omega, \mathcal{P}(\omega), \in, 0, 1, +, \cdot)^{V}$$

In particular, Projective Determinacy holds in L^{2b}.

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Point: For each formula ϕ for the second order arithmetic, there is a Skolem function f for ϕ which is definable in the second order arithmetic s.t. f is invariant under forcing extensions.

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Point: For each formula ϕ for the second order arithmetic, there is a Skolem function f for ϕ which is definable in the second order arithmetic s.t. f is invariant under forcing extensions.

Theorem

Assuming large cardinals, one can show that L^{2b} is invariant under set forcing extensions, i.e., for any poset P, $(\mathrm{L}^{2b})^V = (\mathrm{L}^{2b})^{V^P}$.

Inner models from logics: L^{nb}

One can define L^{nb} for $n \geq 3$ in the same way as L^{2b}.

Remark

1. L^{nb} is a transitive proper class model of ZF.
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Question

What are the relationships between L^{mb} and L^{nb} for different m and n?

Main Results

Let L^{Ω} be the inner model from Woodin's Ω-logic.

Theorem

Under some assumptions on large cardinals and Woodin's Ω-logic,

1. $$\mathrm{L}^{2b} \subsetneq \mathrm{L}^{3b} = \mathrm{L}^{4b} = \cdots = \mathrm{L}^{nb} = \cdots = \mathrm{L}^{\Omega}.$$

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2. The model L^{Ω} is a transitive model of ZFC+GCH.

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2. The model L^{Ω} is a transitive model of ZFC+GCH.
3. The model L^{Ω} is "very big" w.r.t. inner model theory & descriptive set theory.

Main Results ctd.: More precise statements

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3. The reals in L^{Ω} are exactly those which are Δ^2_1(uB) in a countable ordinal.

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3. The reals in L^{Ω} are exactly those which are $\Delta^2_1(\mathrm{uB})$ in a countable ordinal.
4. L^{Ω} is A-closed for any universally Baire set A which is $\Sigma^2_1(\mathrm{uB})$.

Questions

What kind of large cardinals could exist in L^{Ω}?

Conjecture

There is NO measurable cardinal in L^{Ω}.

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Conjecture

There is NO measurable cardinal in L^{Ω}.

Does $(\mathrm{L}^{\Omega}_{\alpha} \mid \alpha \in \mathrm{On})$ have some kind of condensation property?

Background: Universally Baire sets

Definition

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1. The collection of all uB sets is closed under complements and countable unions, hence every Borel set is universally Baire.
2. Every $\mathbf{\Pi}^1_1$-set of reals is universally Baire.
3. Assuming large cardinals, every definable set of reals in the 2nd-order arithmetic is universally Baire.

Background: $\Sigma^2_1(\mathrm{uB})$

Definition

1. A formula ϕ is $\Sigma^2_1(\mathrm{uB})$ if it is of the form

$$(\exists A\colon \text{universally Baire})\ (\omega, \mathcal{P}(\omega), \in, A, 0, 1, +, \cdot) \models \psi,$$

   where ψ is a 2nd-order formula.
2. A set of reals A is $\Sigma^2_1(\mathrm{uB})$ if it is defined by a $\Sigma^2_1(\mathrm{uB})$ formula.
3. A real $x \subseteq \omega$ is $\Delta^2_1(\mathrm{uB})$ in a countable ordinal if there is a countable ordinal α such that both x and $\omega \setminus x$ and are $\Sigma^2_1(\mathrm{uB})$ with the parameter α.

Universally Baire sets ctd.

Remark

A set of reals A is universally Baire if and only if for any partial order $\mathbb{P}$, there are trees T, U on $\omega \times Y$ for some Y such that

$$A = \mathrm{p}[T] \text{ and } \Vdash_{\mathbb{P}} \text{"}\mathrm{p}[\check{T}] = \mathbb{R} \setminus \mathrm{p}[\check{U}]\text{"}.$$

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Using this fact and the trees, one can canonically interpret a uB set A in a set generic extension $V[G]$ (namely $\text{p}[T]$ in $V[G]$). We write A_G for this interpreted set in $V[G]$.

Background: Closure under universally Baire sets

Definition (A-closure)

Let A be universally Baire. An ω-model M of ZFC is *A-closed* if for any V-generic filter G on a partial order in M,

$$M[G] \cap A_G \in M[G].$$

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2. For an ω-model M of ZFC, the following are equivalent:
    1. M is A-closed for every Π^1_2-set A, and
    2. M is closed under sharps.

Main results stated again

Theorem

Under some assumptions on large cardinals and Woodin's Ω-logic,

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4. L^{Ω} is A-closed for any universally Baire set A which is Σ^2_1(uB).

Background: Ω-logic

Ω-logic: a logic on generic absoluteness

Definition (Ω-validity)

Let ϕ be a Π_2-sentence with a real parameter in set theory.
Then ϕ is *Ω-valid* if ϕ is true in any set forcing extension.

$$\text{Main interest: } 0^{\Omega} = \{\phi \mid \phi \text{ is } \Omega\text{-valid}\}.$$

Background: Ω-logic ctd.

Example

1. (Shoenfield) Any Π^1_2-sentence true in V is Ω-valid.

Background: Ω-logic ctd.

Example

1. (Shoenfield) Any Π^1_2-sentence true in V is Ω-valid.
2. If $V = \mathrm{L}$, then the Π^1_3-sentence "Every real is constructible" is *not* Ω-valid while it is true in $V (= \mathrm{L})$.

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4. (Steel) Strong forcing axioms such as PFA impliy the same above.

Strong axioms of infinity give us more statements in 0^Ω.

Background: Ω-provability

Definition

Let ϕ be a Π_2-sentence with a real parameter in set theory.
Then ϕ is Ω-provable if there is a universally Baire set A such that

$$(\forall M \text{ c.t.m. of ZFC}) \text{ if } M \text{ is } A\text{-closed, then } M \models \phi.$$

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Example

Assuming large cardinals, any statement in the 2nd-order arithmetic true in V is Ω-provable.

Background: Ω-Conjecture

Definition

Ω-Conjecture with real parameters states that ϕ is Ω-valid iff ϕ is Ω-provable for all ϕ.

Background: the effect of Ω-Conjecture

With Ω-Conjecture, one can reduce an Ω-valid Π_2 statement to a Σ_1^2 (uB)

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1. All the reals in the mice known to exist so far are Σ^2_1(uB) in a countable ordinal.

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Remark

1. All the reals in the mice known to exist so far are Σ_1^2(uB) in a countable ordinal.
2. If M is A-closed for every A which is universally Baire and Σ_1^2(uB), then M is closed under all the mouse operators known to exist so far.

Background: AD$^+$-Conjecture

We would like to make Ω-valid statements definable in $H_{\mathfrak{c}^+}$. So we need:

Definition

AD$^+$-Conjecture states the following:
Suppose A, B are sets of reals such that $\mathrm{L}(A, \mathbb{R})$ and $\mathrm{L}(B, \mathbb{R})$ are models of AD$^+$.
Assume also that every set of reals in $\mathrm{L}(A, \mathbb{R}) \cup \mathrm{L}(B, \mathbb{R})$ is ω_1-universally Baire.
Then either $\boldsymbol{\Delta}^{2\,\mathrm{L}(A,\mathbb{R})}_1 \subseteq \boldsymbol{\Delta}^{2\,\mathrm{L}(B,\mathbb{R})}_1$ or vice versa.

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Then either $\boldsymbol{\Delta}_1^{2^{\mathrm{L}(A,\mathbb{R})}} \subseteq \boldsymbol{\Delta}_1^{2^{\mathrm{L}(B,\mathbb{R})}}$ or vice versa.

Theorem (Woodin)

1. Suppose there are a proper class of Woodin cardinals and assume that AD^+-Conjecture holds. Then the set of Ω-provable statements is definable in $H_{\mathfrak{c}^+}$.
2. MM implies that AD^+-Conjecture holds.

Inner models from logics: Back to Theorem...

Theorem

Suppose there are a proper class of Woodin cardinals. Assume that the Ω-Conjecture with real parameters and AD^+-Conjecture hold in any set generic extension. Then

$$\mathrm{L}^{3b} = \mathrm{L}^{4b} = \cdots = \mathrm{L}^{nb} = \cdots .$$

For the proof, we introduce L^{Ω} from Ω-logic and show that $Def_{3b} = Def_{\Omega}$.

Inner models from logics: Def_Ω

Definition

Let ϕ be a Σ_2 formula and ψ be a Π_2 formula in the language of set theory. We say (ϕ, ψ) is a Δ_2^{ZFC}-pair if

$$\mathsf{ZFC} \vdash \text{“}(\forall \vec{x})\ \phi(\vec{x}) \leftrightarrow \psi(\vec{x})\text{”}.$$

Inner models from logics: Def_Ω

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Definition

Let A be a first-order structure, $\vec{a} \in A^{<\omega}$, and (ϕ, ψ) be a Δ_2^{ZFC}-pair. Then the triple $(\phi, \psi, \vec{a})$ is suitable to A if for any element x of A, either $\psi[x, \vec{a}, A]$ or $\neg\phi[x, \vec{a}, A]$ is Ω-valid.

Inner models from logics: L^{Ω}

Definition

1. Let $(\phi, \psi, \vec{a})$ be suitable to A. Then a set $X \subseteq A$ is Ω-definable via $(\phi, \psi, \vec{a})$ if $X = \{x \in A \mid (\forall P\colon \text{poset})\ V^P \vDash \phi[x, \vec{a}, A]\}$.
2. $\mathrm{Def}_{\Omega}(A)$ is the collection of Ω-definable subset of A via some $(\phi, \psi, \vec{a})$ suitable to A.

One can define $\mathrm{L}^{\Omega}_{\alpha}$ and L^{Ω} in the same way as before.