The Uniform Martin's Conjecture and the Wadge Degrees

Takayuki Kihara

Joint Work with Antonio Montalbán

Department of Mathematics, University of California, Berkeley, USA

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Under some set theoretic hypothesis, we show that:

There is a natural one-to-one correspondence between the "*natural*" many-one degrees and the Wadge degrees.

Definition

1. Let $A, B \subseteq \omega$. A is many-one reducible to B if there is a computable function $\Phi : \omega \to \omega$ such that
$$(\forall n \in \omega)\ n \in A \iff \Phi(n) \in B.$$
2. Let $A, B \subseteq \omega^\omega$. A is Wadge reducible to B if there is a continuous function $\Psi : \omega^\omega \to \omega^\omega$ such that
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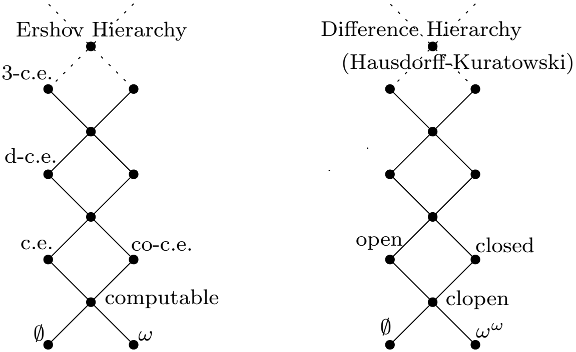
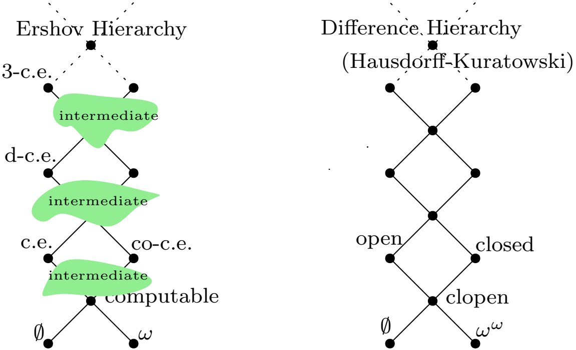


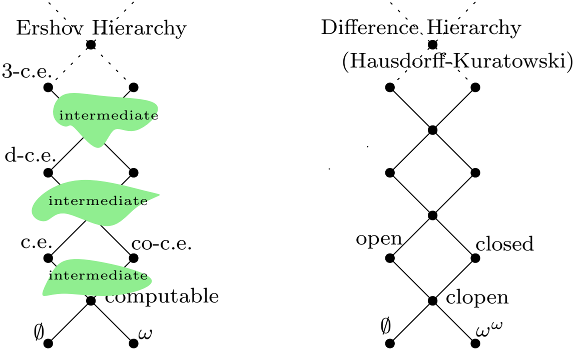
Many-one degrees versus Wadge degrees



Many-one degrees versus Wadge degrees

The structure of the many-one degrees is very complicated:

- There are continuum-size antichains, every countable distributive lattice is isomorphic to an initial segment, etc.
- (Nerode-Shore 1980) The theory of the many-one degrees is computably isomorphic to the true second-order arithmetic.



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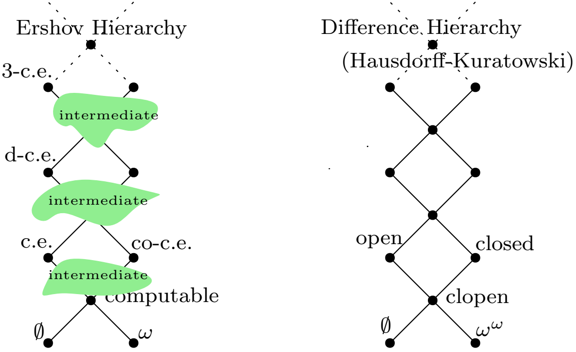

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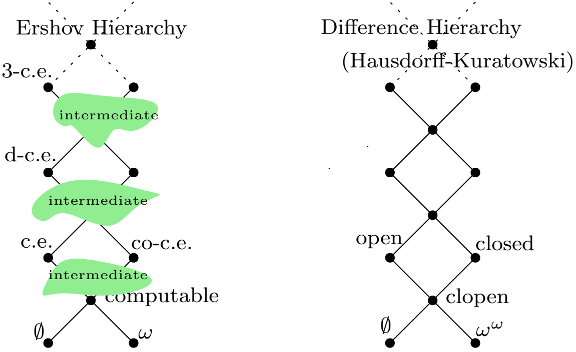

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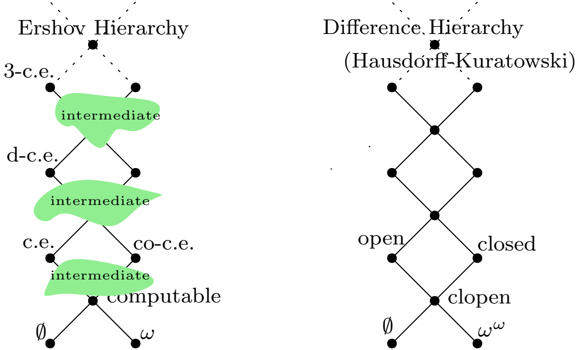

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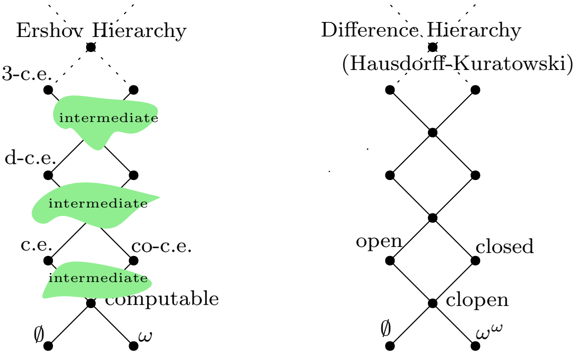

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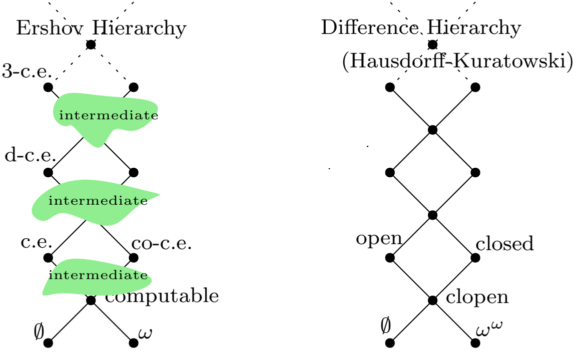

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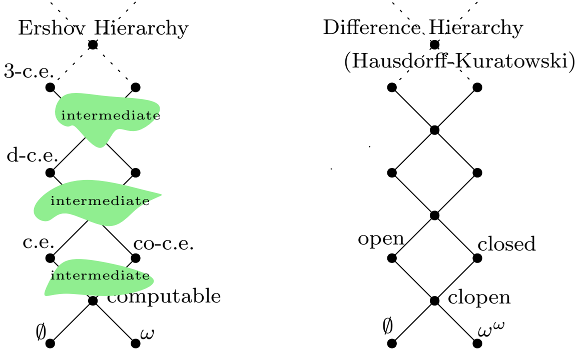

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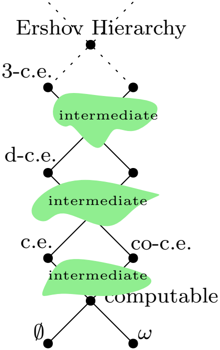

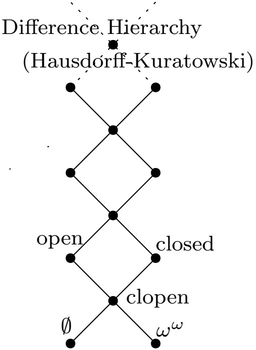

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  - (Relativizability) It is a function $f : 2^\omega \to 2^\omega$.
  - (Degree-Invariance) $X \equiv_T Y$ implies $f(X) \equiv_T f(Y)$.
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- (The Martin Conjecture) There is no intermediate natural Turing degree at each level in the following sense:
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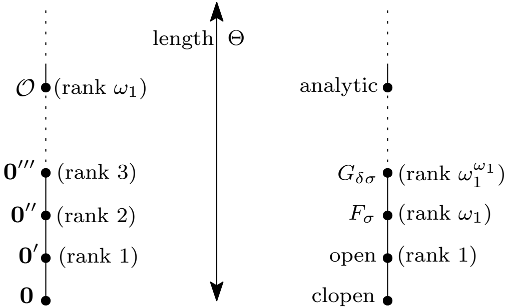
  - Every Degree invariant functions function is either constant or increasing.
  - Degree invariant increasing functions are well-ordered,
  - and each successor rank is given by the Turing jump.
- (Steel 1982; Slaman-Steel 1988) The Martin Conjecture holds true for uniformly degree invariant functions.



Natural Turing degrees and Wadge degrees

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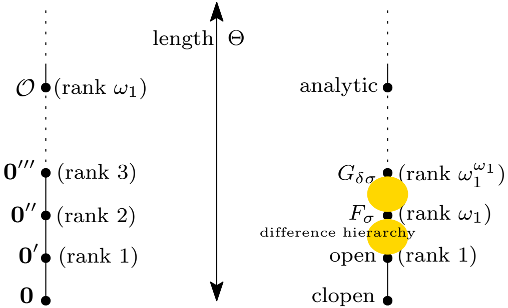
- (Becker 1988) Indeed, uniformly degree invariant increasing functions form a well-order of type $\boldsymbol{\Theta}$.



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(Hypothesis) Natural degrees are relativizable and degree-invariant.

Definition

$f : 2^\omega \to 2^\omega$ is uniformly $(\equiv_T, \equiv_m)$-invariant if there is a function $u : \omega^2 \to \omega^2$ such that for all $X, Y \in 2^\omega$,

$$X \equiv_T Y \text{ via } (i, j) \implies f(X) \equiv_m f(Y) \text{ via } u(i, j).$$

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Definition

Given $f, g : 2^\omega \to 2^\omega$, we say that f is many-one reducible to g on a cone (written as $f \leq_{\mathrm{m}}^{\nabla} g$) if

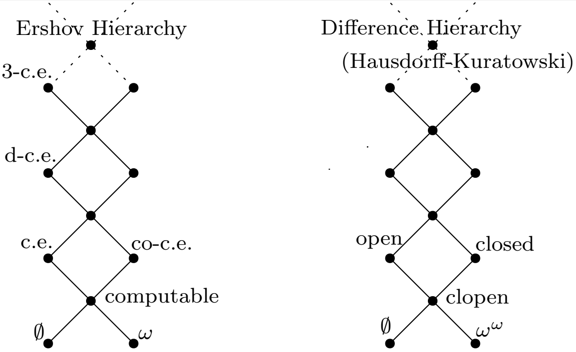
$$(\exists C \in 2^\omega)(\forall X \geq_T C)\ f(X) \leq_m^C g(X).$$

Here $\leq_m^C$ is many-one reducibility relative to C.

Theorem (K.-Montalbán)

(**ZF** + **DC**$_{\mathbb{R}}$ + **AD**) The $\equiv_{\mathbf{m}}^{\nabla}$-degrees of uniformly invariant functions are isomorphic to the Wadge degrees.

(Cor.) The $\equiv_{\mathbf{m}}^{\nabla}$-degrees of UI functions form a semi-well-order of length $\mathbf{\Theta}$.



Natural many-one degrees and Wadge degrees

Theorem (K.-Montalbán)

($\mathbf{ZF} + \mathbf{DC}_{\mathbb{R}} + \mathbf{AD}$) The $\equiv^{\nabla}_{\mathbf{m}}$-degrees of uniformly invariant functions are isomorphic to the Wadge degrees.

Our proof involves heavy game-theoretic arguments,
— and surprisingly, it makes use of the degree-theoretic analysis of thin $\mathbf{\Pi}^0_1$ classes.

Theorem (K.-Montalbán)

($\mathbf{ZF} + \mathbf{DC}_{\mathbb{R}} + \mathbf{AD}$) The $\equiv^{\nabla}_{\mathbf{m}}$-degrees of uniformly invariant functions are isomorphic to the Wadge degrees.

Under the stronger hypothesis $\mathbf{AD}^+$, our result is generalized to $\mathcal{Q}$-valued functions for any better-quasi-order (BQO) $\mathcal{Q}$.

Let $\mathcal{Q}$ be a quasi-order.

1. $\mathcal{Q}$ is a well-quasi-order (WQO) if it has no infinite decreasing seq. and no infinite antichain. It is equivalent to saying that
$$(\forall f : \omega \to \mathcal{Q})(\exists m < n)\ f(m) \leq_{\mathcal{Q}} f(n).$$
2. (Nash-Williams 1965) $\mathcal{Q}$ is a better-quasi-order (BQO) if
$$(\forall f : [\omega]^{\omega} \to \mathcal{Q} \text{ continuous})(\exists X \in [\omega]^{\omega})\ f(X) \leq_{\mathcal{Q}} f(X^{-}).$$
where X^{-} is the shift of X, that is, $X^{-} = X \setminus \{\min X\}$.

BQO $\Longrightarrow$ WQO.
(Example) A finite quasi-order is a BQO. A well-order is a BQO.

- Every $A \in 2^\omega$ is called a decision problem.
- We call $A \in Q^\omega$ a Q-valued problem.
- One can introduce the notions of many-one degrees of Q-valued problems, uniformly invariant Q-valued problems, etc.
- The study of the Wadge degrees of Q-valued functions $A : \omega^\omega \to Q$ provides a new insight even on the Wadge degrees of subsets of ω^ω.

Definition

Let Q be a quasi-order.

1. Let $A, B : \omega \to Q$. A is many-one reducible to B if there is a computable function $\Phi : \omega \to \omega$ such that
$$(\forall n \in \omega)\ A(n) \leq_Q B \circ \Phi(n).$$
2. Let $A, B : \omega^\omega \to Q$. A is Wadge reducible to B if there is a continuous function $\Psi : \omega^\omega \to \omega^\omega$ such that
$$(\forall x \in \omega^\omega)\ A(x) \leq_Q B \circ \Psi(x).$$

- (van Engelen-Miller-Steel 1987) If Q is BQO, the Borel Q-Wadge degrees form a BQO as well.
- For a well-order Q, the Q-Wadge degrees have been studied by Steel (1980s?), Duparc (2003), Block (2014) and others.
- For a finite discrete order Q, the Q-Wadge degrees have been studied by Hertling (1996), Selivanov (2007) and others.

Theorem (K.-Montalbán)

($\mathbf{AD}^+$) Let Q be BQO.

The $\equiv^{\nabla}_{\mathbf{m}}$-degrees of uniformly invariant Q-valued problems are isomorphic to the Wadge degrees of Q-valued functions on ω^ω.

$\mathbf{AD}^+ = \mathbf{DC}_{\mathbb{R}} +$ "every set of reals is ∞-Borel" $+$ "$< \Theta$-Ordinal Determinacy".

Complete description of the Wadge degrees of Borel functions

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Complete description of the Wadge degrees of Borel functions

- "Natural many-one degrees" are exactly the Wadge degrees.
- — Does there exist an easy description of the $\mathcal{Q}$-valued Wadge degrees?
- The complete description of the Wadge degrees of Borel subsets of ω^ω is given by Louveau-Saint Raymond, Duparc and others (using Boolean operations, exotic operations, ..., sometimes hard to understand).
- Selivanov gave a tree-representation of the Wadge degrees of $\mathbf{\Delta}^0_2$-measurable $\boldsymbol{k}$-partitions, and so on.
- We extend their works to $\mathcal{Q}$-valued functions.

- $\mathbf{Tree}(S)$: The set of all S-labeled well-founded countable trees.
- $^{\sqcup}\mathbf{Tree}(S)$: The set of all S-labeled countable forests with no infinite chain.

Theorem (K.-Montalbán)

Let $\mathcal{Q}$ be a BQO.

- The $\mathcal{Q}$-Wadge degrees of $\utilde{\mathbf{\Delta}}^0_2$-functions $\simeq$ $^{\sqcup}\mathbf{Tree}(\mathcal{Q})$.
- The $\mathcal{Q}$-Wadge degrees of $\utilde{\mathbf{\Delta}}^0_3$-functions $\simeq$ $^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathcal{Q}))$.
- The $\mathcal{Q}$-Wadge degrees of $\utilde{\mathbf{\Delta}}^0_4$-functions $\simeq$ $^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(\mathcal{Q})))$.
- The $\mathcal{Q}$-Wadge degrees of $\utilde{\mathbf{\Delta}}^0_5$-functions $\simeq$ $^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(\mathcal{Q}))))$.
- and so on...

- Wadge degrees of **2**-valued Borel functions $\approx$ ordinals.
- Wadge degrees of $\mathcal{Q}$-valued Borel functions $\approx$ (nested) $\mathcal{Q}$-trees.
- This tree-representation gives a very clear description of the Wadge degrees with a (relatively) simple and easy proof (even for general $\mathcal{Q}$), — so I have an impression that the tree-representation is *the* correct way of describing Wadge degrees of Borel sets/functions.

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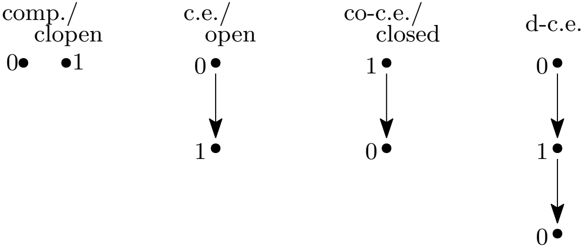
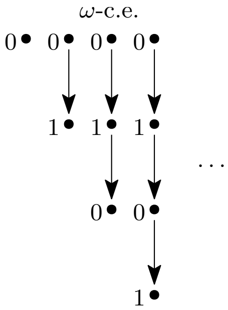
  - Our article on tree representation for general Q consists of **27** pages including introduction etc.; the proof itself is only about **10** pages.

Tree/Forest-representation of various Δ^0_2 sets:



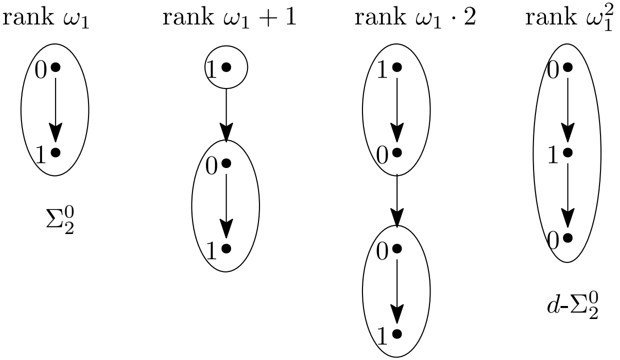
- (computable/clopen) Given an input x, effectively decide $x \notin A$ (indicated by 0) or $x \in A$ (indicated by 1).
- (c.e./open) Given an input x, begin with $x \notin A$ (indicated by 0) and later x can be enumerated into A (indicated by 1).
- (co-c.e./closed) Given an input x, begin with $x \in A$ (indicated by 1) and later x can be removed from A (indicated by 0).
- (d-c.e.) Begin with $x \notin A$ (indicated by 0), later x can be enumerated into A (indicated by 1), and x can be removed from A again (indicated by 0).

Forest-representation of a complete ω-c.e. set:



(ω-c.e.) The representation of "ω-c.e." is a forest consists of linear orders of finite length (a linear order of length $\boldsymbol{n+1}$ represents "$\boldsymbol{n}$-c.e.").

- Given an input $\boldsymbol{x}$, effectively choose a number $\boldsymbol{n} \in \omega$ giving a bound of the number of times of mind-changes until deciding $\boldsymbol{x \in A}$.



Tree/Forest-representation of $\underset{\sim}{\mathbf{\Delta}}^0_3$ sets

The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_3$ sets are exactly those represented by
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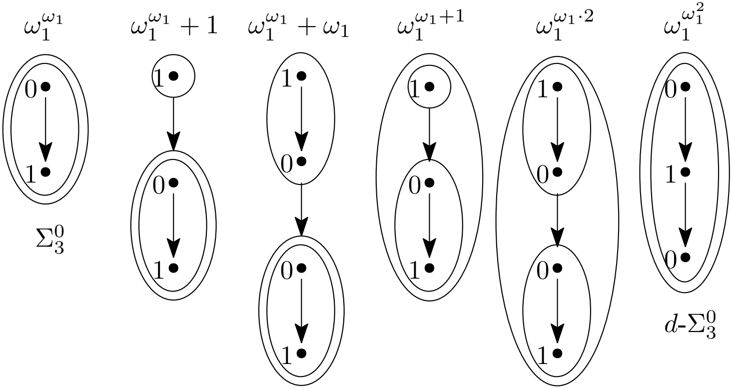

Tree/Forest-representation of $\mathbf{\Delta}_4^0$ sets

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- $\mathbf{Tree}(S)$: The set of all S-labeled well-founded countable trees.
- ${}^{\sqcup}\mathbf{Tree}(S)$: The set of all S-labeled countable forests with no infinite chain.

Theorem

Let Q be a BQO.

- The Q-Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_2$-functions $\simeq {}^{\sqcup}\mathbf{Tree}(Q)$.
- The Q-Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_3$-functions $\simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(Q))$.
- The Q-Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_4$-functions $\simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q)))$.
- The Q-Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_5$-functions $\simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q))))$.
- and so on...

Theorem (K.-Montalbán [1])

1. ($\mathbf{AD} + \mathbf{DC}_{\mathbb{R}}$) There is an isomorphism between the $\equiv^{\nabla}_{\mathbf{m}}$-degrees of UI decision problems and the Wadge degrees of subsets of ω^ω.
2. ($\mathbf{AD}^+$) For any BQO Q, there is an isomorpshim between the $\equiv^{\nabla}_{\mathbf{m}}$-degrees of UI Q-valued problems and the Wadge degrees of Q-valued functions on ω^ω.

$\mathbf{AD}$ = The Axiom of Determinacy (every set of reals is determined).

$\mathbf{DC}_{\mathbb{R}}$ = The Dependent Choice on $\mathbb{R}$.

$\mathbf{AD}^+$ = $\mathbf{DC}_{\mathbb{R}}$ + "every set of reals is ∞-Borel" + "$< \Theta$-Ordinal Determinacy".

Theorem (K.-Montalbán [2])

$$(\underset{\sim}{\mathbf{\Delta}}^0_{1+\xi}(\omega^\omega, Q), \leq_w) \simeq ({}^{\sqcup}\mathbf{Tree}^{\xi}(Q), \trianglelefteq).$$

[1] T. Kihara and A. Montalbán, The uniform Martin's conjecture for many-one degrees, submitted (`arXiv:1608.05065`).

[2] T. Kihara and A. Montalbán, On the structure of the Wadge degrees of BQO-valued Borel functions, in preparation.

Definition

1. We say that $A \subseteq [\omega]^\omega$ is Ramsey if there is $X \in [\omega]^\omega$ such that either $[X]^\omega \subseteq A$ or $[X]^\omega \cap A = \emptyset$.
2. **Γ-Det** is the hypothesis "*every* Γ *set of reals is determined*".
3. **Γ-Ramsey** is the hypothesis "*every* Γ *set of reals is Ramsey*".

Remark

What we really need is the hypothesis

"every Γ set of reals is completely Ramsey"

(i.e., every Γ set has the Baire property w.r.t. Ellentuck topology)

but for most natural pointclasses Γ, this hypothesis is known to be equivalent to **Γ-Ramsey** (Brendle-Löwe (1999)).

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- (Martin 1975) **ZF + DC** $\vdash$ **Borel-Det**.
- (Galvin-Prikry 1973; Silver 1970) **ZF + DC** $\vdash$ $\utilde{\mathbf{\Sigma}}^1_1$-**Ramsey**.
- (Harrington-Kechris 1981) **PD** implies **Projective-Ramsey**.
  - Indeed, they showed that $\utilde{\mathbf{\Delta}}^1_{2n+2}$-**Det** implies $\utilde{\mathbf{\Pi}}^1_{2n+2}$-**Ramsey**.
  - (Fang-Magidor-Woodin 1992) $\utilde{\mathbf{\Sigma}}^1_1$-**Det** implies $\utilde{\mathbf{\Sigma}}^1_2$-**Ramsey**.
- (Open Problem) Does **AD** imply that every set of reals is Ramsey?
- (Solovay; Woodin) $\mathbf{AD}^+$ implies that every set of reals is Ramsey.
  - $\mathbf{AD}^+ = \mathbf{DC}_{\mathbb{R}} +$ "every set of reals is ∞-Borel" + "$< \Theta$-Ordinal Determinacy".

Why **Γ-Ramsey**? Because we need the following lemma:

Lemma ($\mathbf{ZF} + \mathbf{DC}_{\mathbb{R}} + \mathbf{\Gamma}$-Det $+ \mathbf{\Gamma}$-Ramsey)

Let Q be a BQO.

1. The Q-Wadge degrees of **Γ**-functions form a BQO.
2. A Q-Wadge degree of **Γ**-functions is self-dual if and only if it is σ-join-reducible.

Proof

1. Louveau-Simpson (1982) showed that if a function f from $[\omega]^\omega$ into a metric space has the Baire property w.r.t. Ellentuck topology, then there is an infinite set X such that the restriction $f \upharpoonright [X]^\omega$ is continuous w.r.t. Baire topology. Combine this result with van Engelen-Miller-Steel (1987).
2. For $Q = (2, =)$, it has been shown by Steel-van Wesep (1978) (without **Γ-Ramsey**). Recently Block (2014) introduced the notion of vsBQO and extended the Steel-van Wesep Theorem to vsBQO. Analyze Block's proof, and combine it with Louveau-Simpson (1982).